Gauge Theory Wilson Loops and Conformal Toda Field Theory

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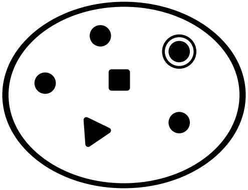
A large class of 4-dimensional $\mathcal{N} = 2$ gauge theories

Gaiotto has constructed a large class of four dimensional $\mathcal{N} = 2$ gauge theories that describe the low energy dynamics of a stack of N M5-branes compactified on a punctured Riemann surface $C_{(f_a),g}$.



A four dimensional gauge theory $\mathcal{T}_{(f_a),g}$ is characterized by the same data labeling the surface

- the genus g
- the number of punctures (f_a). There are different types of puncture and each type is labeled by a Young tableaux with N boxes.

More in details

- the (f_a) punctures encode the flavor symmetry of the gauge theory $\mathcal{T}_{(f_a),g}$
- the different degenerations of the surface $C_{(f_a),g}$ such that it becomes a set of pairs of pants connected by thin tubes are associated to the different S-duality frame of the gauge theory $\mathcal{T}_{(f_a),g}$. The thin tubes connecting the pair of pants are the weakly coupled gauge groups
- the moduli space of $C_{(f_a),g}$ is equal to the parameter space of $\mathcal{T}_{(f_a),g}$

The connection between 4D and 2D physics can be made even more precise.....

From 4D to 2D: the AGT proposal

The partition function of four dimensional gauge theories $\mathcal{T}_{(f_a),g}(A_{N-1})$ defined on S^4 is equivalent to a correlator of two dimensional A_{N-1} Toda field theory defined on $C_{(f_a),g}$

$$Z_{\mathcal{T}_{f,g}} = \langle V_{m_1} \ldots V_{m_f} \rangle_{\text{Toda on } C_{(f_a),g}}$$

- there is one primary for each puncture and the momenta $m_1, \ldots, m_f$ are related to the masses of the hypermultiplets
- different correlators of the same 2D field theory compute the partition function of different 4D gauge theories

Where does it come from?

The partition function for $\mathcal{N} = 2$ gauge theories can be written as

$$Z_{\mathcal{T}_{f,g}} = \int [da] Z_{\text{Nekrasov}}(q, a, m, \epsilon_1, \epsilon_2) \bar{Z}_{\text{Nekrasov}}(\bar{q}, a, m, \epsilon_1, \epsilon_2)$$

- a is a VEV of adjoint scalars in the vector multiplets

Toda Field Theory

A_{N-1} Toda field theory action

$$S_{A_{N-1}} = \int dx^2 \sqrt{g} \left(\frac{1}{8\pi} g^{\alpha\beta} \langle \partial_\alpha \varphi, \partial_\beta \varphi \rangle + \frac{\langle Q, \varphi \rangle}{4\pi} R + \mu \sum_{k=1}^{N-1} e^{b \langle e_k, \varphi \rangle} \right)$$

- $\varphi = \sum_{k=1}^{N-1} \varphi_k e_k$ where e_k is a simple root of A_{N-1} algebra
- $\langle e_i, e_j \rangle = K_{ij}$ is the scalar product in the root space where K_{ij} is the Cartan matrix of the A_{N-1} algebra
- $g_{\alpha\beta}$ and R are the non-dynamical metric and curvature of the Riemann surface
- Q is the background charge and conformal invariance requires

$$Q = q\rho \tag{4.1}$$

where $q = \left(b + \frac{1}{b}\right)$ and ρ is the Weyl vector (i.e. the sum of all the fundamental weights of A_{N-1}).

Besides conformal invariance, A_{N-1} conformal Toda field theory enjoys also higher spin symmetries !!

Reminder

A_{N-1} is the algebra associated to the $SU(N)$ group. It has a linear Dynkin diagram and the Cartan matrix $K_{ij} = \langle e_i, e_j \rangle$ is given by

$$K_{ij} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

$\mathcal{W}_N$ algebra

There are in total $N-1$ holomorphic currents $W^{(i+1)}$ $(i=1,\dots,N-1)$

- Each current has conformal dimension $(i+1)$
- $W^{(2)}=T$ is the stress tensor with conformal dimension 2
- The Laurent expansions of the currents are

$$W^{(i+1)}(z)=\sum_n \frac{W_n^{(i+1)}}{z^{n+i+1}} \tag{4.2}$$

The $N-1$ symmetry currents form a $\mathcal{W}_N$ algebra, a consistent extension of the Virasoro symmetry.

- when $N=2$, the algebra $\mathcal{W}_2$ is the Virasoro algebra. Indeed the A_1 Toda field theory is Liouville theory, a theory that posses Virasoro invariance.

The system includes also antiholomorphic currents so that the total symmetry is $\mathcal{W}_N \times \overline{\mathcal{W}}_N$

$\mathcal{W}_N$ primaries

The $\mathcal{W}_N$ primary fields V, are defined such that

$$W_0^{(i+1)}V = w^{(i+1)}V, \qquad W_n^{(i+1)}V = 0 \qquad \text{for} \qquad n > 0 \tag{4.3}$$

Descendant fields are obtained acting on the primaries with the operators $W_{-n}^{(i+1)}$ where $n > 0$. For instance, for $\mathcal{W}_3$ algebra, at level-1 in the Verma module of V there are $W_{-1}^{(2)}V$ and $W_{-1}^{(3)}V$.

In Toda field theory the primaries are realized as exponential fields

$$V_\alpha = e^{\langle \alpha, \varphi \rangle}$$

- α is a vector in the root space of the A_{N-1} algebra
- the conformal dimension is $\Delta(\alpha) \equiv w^{(2)}(\alpha) = \frac{1}{2}\langle \alpha, 2Q - \alpha \rangle$

Toda degenerate fields

A degenerate field is a field that has (at least) a null descendant. The Verma module is thus a reducible representation of the $\mathcal{W}_N$ A completely degenerate

field is a field that has N-1 null descendants. They are characterized by a vector α given by

$$\alpha = -b\Omega_1 - \frac{1}{b}\Omega_2$$

- Ω_1 and Ω_2 are two highest weights of finite dimensional representations of the algebra A_{N-1}.

In the OPE of completely degenerate fields with a generic primary appear only a finite set of primaries

$$V_{-b\Omega_1-\frac{1}{b}\Omega_2} \cdot V_\alpha = \sum_{s,t} [V_{\alpha'_{s,t}}]$$

where $\alpha'_{s,t} = \alpha - bh_s^{\Omega_1} - \frac{1}{b}h_t^{\Omega_2}$ and h_s^{Ω} are the weights of the representation of A_{N-1} that has Ω as highest weight.

A_{N-1} Toda theory with $N > 2$ are much less understood than Liouville theory (i.e. A_1 Toda)

- in the general case it is not possible to decompose the higher point correlation functions in terms of three point functions of $\mathcal{W}_N$ primary fields and $\mathcal{W}_N$ conformal blocks
- the three point function of primary fields is known exactly only when one of the insertion is a certain degenerate field

The four point function is known exactly only when two of the insertion are completely degenerate fields. This is exactly the correlators that we need to study Wilson loops operator !!

Four point correlation function

The four point correlation function with two degenerate insertions $V_{-b\omega_1}$ and $V_{-b\omega_{N-1}}$ results

$$\langle V_{\alpha_1}(0)V_{-b\omega_1}(z,\bar{z})V_{-b\omega_{N-1}}(1)V_{\alpha_2}(\infty)\rangle = |z|^{2b\langle\alpha_1,h_1\rangle}|1-z|^{\frac{-2b^2}{N}}G(z,\bar{z})$$

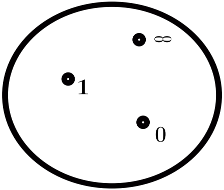
where $G(z,\bar{z})$ satisfies the generalized hypergeometric differential equation in each of the two complex variables z and $\bar{z}$

$$D(A_1,\ldots,A_N;B_1,\ldots,B_N)G(z,\bar{z})=0,\quad \bar{D}(A_1,\ldots,A_N;B_1,\ldots,B_N)G(z,\bar{z})=0$$

where

- $D(A_1,\ldots,A_N;B_1,\ldots,B_N)=z(z\partial+A_1)\ldots(z\partial+A_N)-(z\partial+B_1-1)\ldots(z\partial+B_N-1)$
- $A_k=-b^2+b\langle\alpha_1-Q,h_1\rangle+b\langle\alpha_2-Q,h_k\rangle$
- $B_k=1+b\langle\alpha_1-Q,h_1\rangle-b\langle\alpha_1-Q,h_{k+1}\rangle$

The solutions of the differential equation are defined on the Riemann sphere and are singular at $0, 1, \infty$, the positions where we have located three of the fields

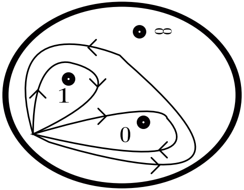


In each punctured neighborhood of the singularities is possible to define N linearly independent solutions

$$\Lambda^{(s)} = (\Lambda_1^{(s)}, \ldots, \Lambda_N^{(s)}) \quad \text{defined in a neighborhood of } 0,$$
$$\Lambda^{(t)} = (\Lambda_1^{(t)}, \ldots, \Lambda_N^{(t)}) \quad \text{defined in a neighborhood of } 1,$$
$$\Lambda^{(u)} = (\Lambda_1^{(u)}, \ldots, \Lambda_N^{(u)}) \quad \text{defined in a neighborhood of } \infty.$$

Monodromy

The solutions can be analytically continued outside the domain of definition and is possible to consider analytical continuations along closed paths. When the path encircles one or more singularities, the vector of solution is linearly transformed by an element of $GL(N, \mathbb{C})$, i.e. a monodromy matrix



Given a vector of N linearly independent solutions, the monodromy matrices computed around the three homotopy classes of the three punctured sphere, $M_{(0)}$, $M_{(1)}$ and $M(\infty)$, form a subgroup of $GL(N, \mathbb{C})$. This is the monodromy group, defined by the following relation

$$M_{(\infty)} M_{(1)} M_{(0)} = 1$$

It is a representation on the linear space of solutions of the first homotopy group of the Riemann sphere with three punctures.

- The monodromy group is invariant under conjugation inside $GL(N, \mathbb{C})$. Given two vector solutions related by the linear transformation $\tilde{\Lambda} = X\Lambda$, the monodromy matrices $\tilde{M} = XMX^{-1}$ are still a representation of the monodromy group.
- The conformal blocks in the s, t, u channel are given by

$$
\begin{aligned}
\mathcal{F}_k^{(s)} &= z^{b\langle\alpha_1,h_1\rangle}(1-z)^{\frac{-b^2}{N}}\Lambda_k^{(s)}, \\
\mathcal{F}_k^{(t)} &= z^{b\langle\alpha_1,h_1\rangle}(1-z)^{\frac{-b^2}{N}}\Lambda_k^{(t)}, \\
\mathcal{F}_k^{(u)} &= z^{b\langle\alpha_1,h_1\rangle}(1-z)^{\frac{-b^2}{N}}\Lambda_k^{(u)}
\end{aligned}
\tag{4.4}
$$

The four point function is obtained considering bilinear combinations of $\mathcal{F}(z)$ and $\bar{\mathcal{F}}(\bar{z})$ that give a single valued function.