WILSON LOOPS: NON LOCAL AND GAUGE INVARIANT OPERATORS

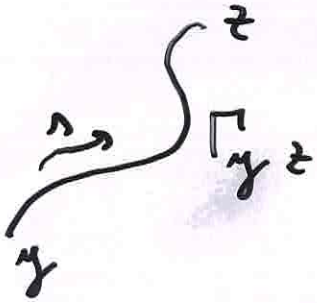
U(1) GAUGE THEORY $\quad [\ A_\mu(x)\ ]$

GAUGE TRANSFORMATION: $A_\mu(x) \xrightarrow{G.T.} A_\mu(x) + \partial_\mu \alpha(x)$

PHASE FACTOR

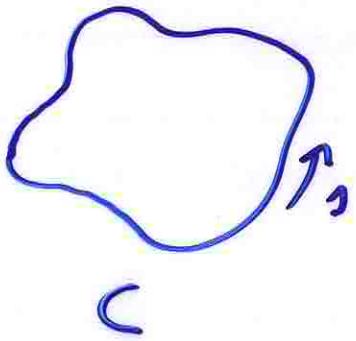
$$U[\Gamma_{yz}] = \exp\left[\, iq \int_{\Gamma_{yz}} dx\, \dot{x}^\mu A_\mu(x) \,\right]$$

$$U[\Gamma_{yz}] \xrightarrow{G.T} e^{iq\alpha(y)}\, U[\Gamma_{yz}]\, e^{iq\alpha(z)}$$



PARALLEL TRANSPORTER

IF $\Gamma = C$



A "LOOP"

$$\longrightarrow \quad W_q(C) = \mathrm{EXP}\left[iq \int_C ds\, \dot{x}^\mu A_\mu(x) \right]$$

GAUGE INVARIANT OPERATOR

- q CHARGE [REPRESENTATION OF $U(1)$]
- C CLOSED CURVE

DESCRIBE THE COUPLING OF THE $U(1)$ GAUGE THEORY WITH A NON DYNAMICAL CHARGED PARTICLE.

U(N) YANG-MILLS

$$W_R(C) = Tr_R \, \mathcal{P} \exp\left[i \int_C ds \, A_\mu^{(x)} \, \dot{x}^\mu \right]$$

- $\mathcal{P}$ PATH ORDERING ($A_\mu = A_\mu^a t^a$

t^a GENERATORS OF U(N))

- Tr TRACE

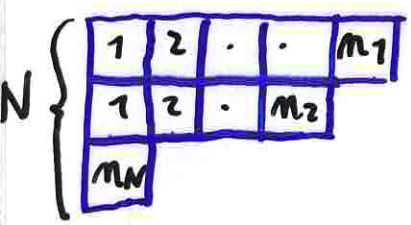
W-L IS CHARACTERIZED BY

C A CLOSED CURVE

R REPRESENTATION OF U(N)

R IS SUMMARIZED BY A YOUNG TABLEAU

$$R = (n_1, n_2, \dots, n_i, \dots, n_N)$$



n_i: BOXES IN THE i-TH ROW

$n_1 \geq n_2 \geq \dots \geq n_N$

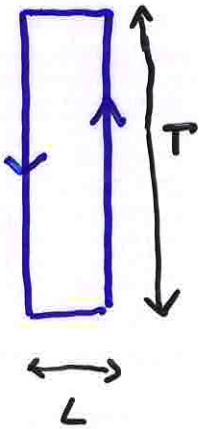
$W_R(C)$ DESCRIBE THE COUPLING OF U(N) YANG-MILLS WITH AN EXTERNAL NON DANYMICAL PARTICLE WITH CHARGE R

E.G. THE EXTERNAL PARTICLE IS A QUARK

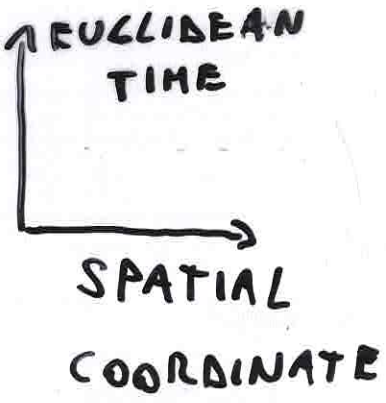
$\longrightarrow$ $R = \square$ FUNDAMENTAL

$$\langle W_{\square}(C_0) \rangle = A(L)\, e^{-T E(L)}$$

$E(L)$: QUARK-ANTIQUARK POTENTIAL

$C_0 =$ 

, $T \to +\infty$



W-L : ORER PARAMETER FOR CONFINEMENT

WHY HOLOGRAPHIC ?

AdS / CFT DUALITY

STATEMENT:

EQUIVALENT

TYPE IIB STRING ON $AdS_5 \times S_5$ WITH $\int_{S_5} F_5 = N$ $\Longleftrightarrow$ $\mathcal{N}=4$ SYM $U(N)$ ON $\mathbb{R}^{3,1}$

PARAMETERS:

RADIUS: $L^4 = 4\pi g_s N \alpha'^2$

Y.M. COUPLING: g_{YM}

$g_s = g_{YM}^2$

T'HOOFT: $\lambda = g_{YM}^2 N$

$N \to +\infty$ LIMIT

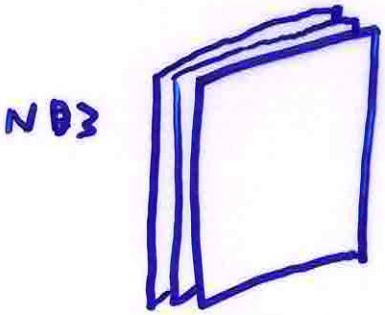
NON INTERACTING STRING THEORY

PLANAR GRAPHS

$\lambda \gg 1$ NO α' CORRECTIONS

$\lambda \ll 1$ PERTURBATIVE REGIME

	0	1	2	3	4	5	6	7	8	9	SPATIAL COORDINATES
N D3	X	X	X	X							



$g_s N \gg 1$

$g_s N \ll 1$

CLOSED STRING DESCRIPTION

OPEN STRING DB SCRIPTION

MALDACENA LIMIT

$AdS_5 \otimes S_5$

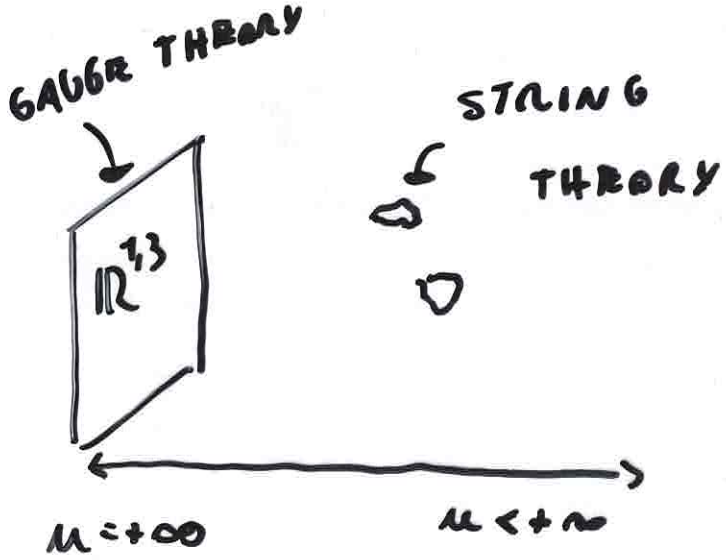
$N = 4$ SYM

HOLOGRAPHY

$$ds^2_{AdS\otimes S} = L^2\left(u^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{du^2}{u^2}\right) + L^2\left(d\theta^2 + \sin^2\theta\, d\Omega_4^2\right)$$

$u \longrightarrow +\infty$ BOUNDARY OF THE SPACE $\mathbb{R}^{1,3}$

THE PHYSICS OF THE BULK RESULTS FROM A HOLOGRAPHIC IMAGE OF THE PHYSICS ON THE BOUNDARY

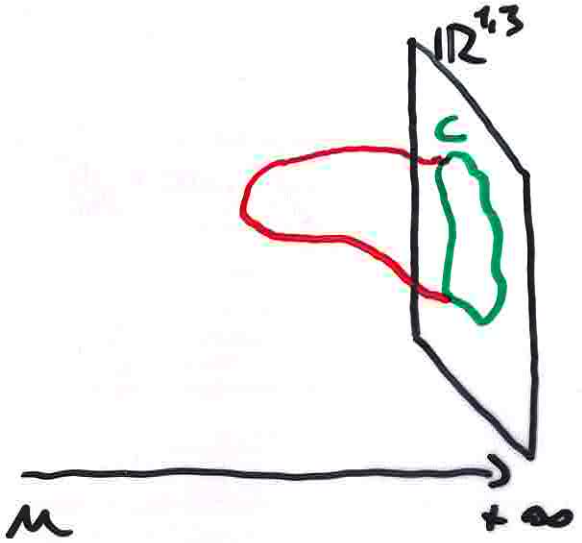


GRAVITATIONAL DESCRIPTION OF WILSON LOOPS

REY, YEE

MALDACENA

$W_{\square}(C)$ IS ASSOCIATED TO A FUNDAMENTAL STRING IN THE BULK ENDING ON C AT THE BOUNDARY OF AdS



$$\langle W_{\square}(C) \rangle \sim e^{-S}$$

→ STRING ACTION, PROPORTIONAL TO THE SURFACE OF THE WORLD-SHEET

FOR NON-CONFORMAL THEORY

$$E(L) \propto L$$

CONFINEMENT

$\mathcal{N}=4$ SYM $\quad [A_\mu, \phi^I, \lambda]$

W.L. DESCRIBE COUPLING OF A SUPERPARTICLE TO SYM

$\longrightarrow$ BOSONIC WILSON LOOPS

$$C \equiv \left(x^\mu(\tau), \eta^I(\tau)\right)$$

$$W_R(C) = \mathrm{Tr}_R \mathcal{P} \exp\left[i \int_C d\tau \left(A_\mu \dot{x}^\mu + \phi_I \dot{\eta}^I\right)\right]$$

$\mathcal{N}=4$ SYM INVARIANT UNDER:

POINCARE SUSY: $\delta_{\varepsilon_1} A_\mu = i \bar{\varepsilon}_1 \gamma_\mu \lambda \qquad \delta_{\varepsilon_1} \phi_I = i \bar{\varepsilon}_1 \gamma_I \lambda$

SUPER CONFORMAL: $\delta_{\varepsilon_2} A_\mu = i \bar{\varepsilon}_2 x^\nu \gamma_\nu \gamma_\mu \lambda \qquad \delta_{\varepsilon_2} \phi_I = i \bar{\varepsilon}_2 x^\nu \gamma_\nu \cdot \gamma_I \lambda$

$\frac{1}{2}$ BPS WILSON LOOPS

W.L. INVARIANT UNDER $\frac{1}{2}$ POINCARE SUSY

AND $\frac{1}{2}$ SUPERCONFORMAL

$\longrightarrow$ SUSY SELECT A PREFERRED CURVE:

$$C = \left(x^0 = t,\ x^i = 0,\ y^I = n^I\right) \qquad (n^I)^2 = 1$$

OPERATOR UNDER EXAM

$$W_R = \mathrm{Tr}_R\, \mathcal{P}\, \exp\left[i \int dt \left(A_0 + \phi\right)\right]$$

SYMMETRIES

SYM	$\frac{1}{2}$ BPS W.L.
$SU(2,2) \simeq SO(2,4)$	$\longrightarrow$ $SU(1,1) \times SU(2)$
$SU(4) \simeq SO(6)$	$\longrightarrow$ $SO(5)$
ϵ_1, ϵ_2	$\longrightarrow$ $\begin{cases} \gamma_{0I} n^I \epsilon_1 = \epsilon_1 \\ \gamma_{0I} n^I \epsilon_2 = -\epsilon_2 \end{cases}$

$PSU(2,2|4) \longrightarrow Osp(4^*|4)$

GRAVITATIONAL DESCRIPTION OF $\frac{1}{2}$ BPS W.L.

WE LOOK FOR PROBES THAT PRESERVE THE $O_{sp}(4^*|4)$ SYMMETRY, I.E. THE SAME AS $\frac{1}{2}$ BPS W.L.

$R = \square$ FUNDAMENTAL REP.

→ FUNDAMENTAL STRING DESCRIBED BY $X^\mu = X^\mu(\sigma^0, \sigma^1)$

$x^0 = \sigma^0$ $u = \sigma^1$ $x^i = 0$ $x^I = m^I$

SPANS AdS_2 ↓ SITS AT $x^i = 0$ ↓ SITS AT A POINT IN S^5 ↓

→ $SU(1,1) \otimes SU(2) \otimes SO(5)$

→ SAME CONDITION ON ϵ_1, ϵ_2 AS W.L.

$O_{sp}(4^*|4)$ SYMMETRY

→ AT THE BOUNDARY THE STRING ENDS ON $x^0 = t$

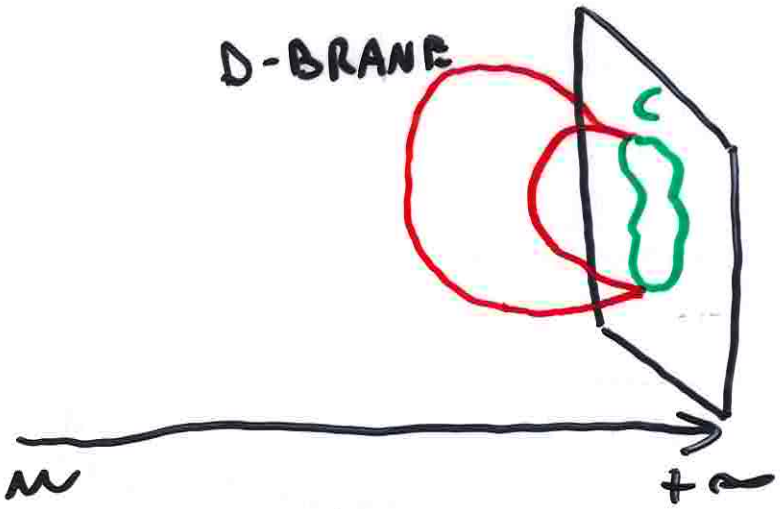
HIGHER REPRESENTATIONS

↓

MULTIPLE (k) COINCIDENT FUNDAMENTAL STRINGS

↓

DESCRIPTION IN TERMS OF A SINGLE D-BRANE WITH (k) UNIT OF FUNDAMENTAL STRING CHARGE



DS_K - BRANE GIANT WILSON LOOP

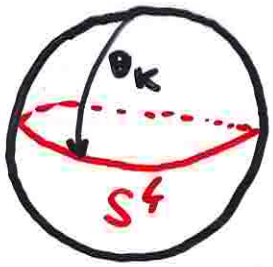
$X^M = X^M(\sigma^0, \sigma^1, \ldots \sigma^S)$ EMBEDDING IN $AdS_5 \otimes S_5$

$$\begin{array}{ccc} AdS_2 & \otimes & S_4 \\ \cap & & \cap \\ AdS_5 & \otimes & S_5 \end{array}$$ EMBEDDING

→ $SU(1,1) \times SU(2) \times SO(5)$ SYMMETRY

→ SAME $\frac{1}{2}$ BPS SUSY

EMBEDDING IN S^5



⇝ θ_K LATITUDE $\in [0, \pi]$

⇝ $\theta_0 = 0$ $\theta_N = \pi$

⇝ INCREASE WITH K

$K \leq N$ MAX NUMBER OF FUNDAMENTAL CHARGES

$DS_K \longleftrightarrow W$ K

$D3_K$ – BRANE DUAL GIANT WILSON LOOP

$$x^M = x^M(\sigma^0, \sigma^1, \sigma^2, \sigma^3)$$

EMBEDDING

$AdS_2 \otimes S_2$ POINT

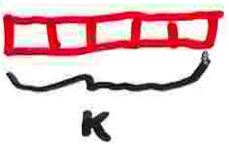
$\cap$ $\cap$

AdS_5 S^5

$\rightarrow$ $SO_{3p}(4^*|4)$ SYMMETRY

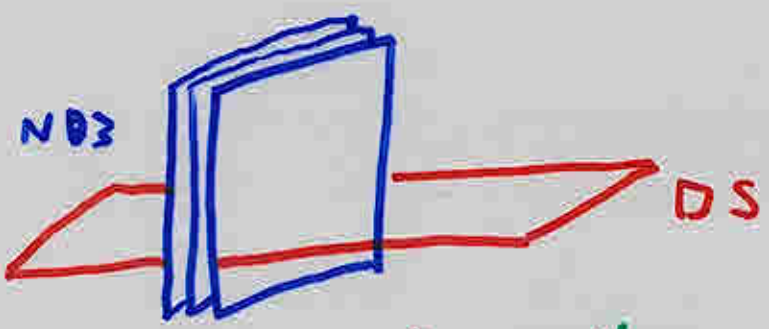
ARBITRARY NUMBER (K) OF FUNDAMENTAL CHARGES

$D3_K$ $\longleftrightarrow$ W



DS_k GIANT WILSON LOOP

	0	1	2	3	4	5	6	7	8	9	
N D3	X	X	X	X							SPATIAL COORDINATES
DS X					X	X	X	X	X		



$g_s N \gg 1$ $g_s N \ll 1$

CLOSED STRING DESCRIPTION

OPEN STRING DESCRIPTION

MALDACENA LIMIT

$AdS_5 \otimes S_5$

$N = 4$ SYM

$\oplus$

DS_k GIANT W.L.

PROBE EMBEDDED IN $AdS \times S$

$\oplus$

→ DEFECT FIELD THEORY (NS-D, D-NS)

LOW ENERGY FIELD THEORY D-S

DEFECT FIELD THEORY

$$S = S_{N=4} + \int dt \left(i\chi^+ \partial_t \chi + \chi^+ (A_0 + \phi)\chi \right)$$

↳ IN THE FUND. REP. OF U

$$+ \; S_{EXTRA}$$

NON DYNAMICAL D.O.F $(\tilde{A})$

$$S_{EXTRA} = \int dt \left(\chi^+ \tilde{A}_0 \chi - k\tilde{A}_0 \right)$$

ENCODE THE CHOICE OF K

WE NOW INTEGRATE OUT THE D.O.F. ASSOCIATED TO DS_K.

GAUGE CHOICE : $A_0 + \phi = \text{DIAG}(w_1, \dots w_N)$

E.O.M. FOR χ's : $(i\partial_t + w_i)\chi_i = 0$

$$Z = \int [D\chi][D\chi^+][DA_0] \; e^{i(S + S_{EXTRA})}$$

INTEGRATING ONLY THE χ, χ^+ $0 < t < \beta$ AND IGNORING S_{EXTRA}

$$Z^* = e^{iS_{N=4}} \cdot \prod_{i=1}^{N} (1 + x_i) \qquad x_i = e^{i\beta}$$

EXPANDING

$$\prod_{i=1}^{N} (1 + x_i) = \sum_{\ell=0}^{N} E_\ell(x_1, \ldots x_\ell)$$

WHERE $E_\ell = Tr_{\{\}_\ell} e^{i\beta w_i} =$

$= W_{\{\}_\ell}$

THUS:

$$Z^* = e^{iS_{N=4}} \sum_{\ell=0}^{N} W_{\{\}_\ell}$$

S_{EXTRA}

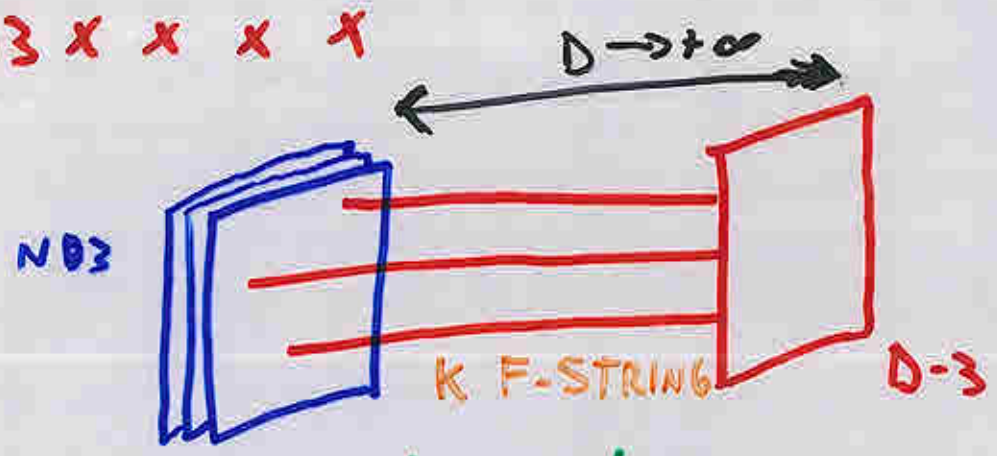
INTEGRATING $\tilde{A}_0 \rightsquigarrow \sum_{i=1}^{N} \chi_i^+ \chi_i = K$

$\longrightarrow \ell = K$

$DS_K \longleftrightarrow Z = e^{iS_{N=4}} W_{\{\}_K}$

$D3_K$ DUAL GIANT WILSON LOOP

	0	1	2	3	4	5	6	7	8	9	
N D3	x	x	x	x							SPATIAL COORDINATES
D3	x	x	x	x							



$g_s N \gg 1$

$g_s N \ll 1$

CLOSED STRING DESCRIPTION

OPEN STRING DESCRIPTION

MALDACENA LIMIT

$AdS_5 \otimes S_5$

⊕

$\rightsquigarrow AdS_2 \otimes S_2 \subset AdS_5$

PROBE D-3

$\mathcal{N}=4$ SYM

⊕

$\rightsquigarrow$ NON RELATIVISTIC W-BOSONS

$\rightsquigarrow$ W-BOSONS INSERTIONS IN THE PATH INTEGRAL

LOW ENERGY FIELD THEORY D3

CONSIDER $U(N+1)$ SYM IN THE COULOMB BRANCH

$$\langle \phi \rangle = \begin{vmatrix} 0 & 0 \\ 0 & D \end{vmatrix}$$

↳ DISTANCE BETWEEN TWO STACKS OF BRANA

$$U(N+1) \longrightarrow U(N) \times U(1)$$

$$\hat{A}_\mu = \begin{vmatrix} A_\mu & w_\mu \\ w_\mu^+ & \tilde{A}_\mu \end{vmatrix}$$

↳ W-BOSONS, TRANSFORM IN THE FUNDAMENTAL OF $U(N)$

D IS THE MASS OF W-BOSONS

$D \longrightarrow +\infty$ IS A NON RELATIVISTIC LIMIT

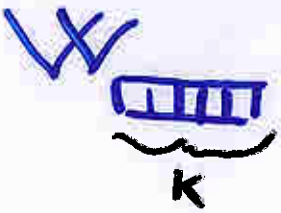
$$S = S_{N=4} + \int i\chi^+ \partial_t \chi + \chi^+ (A_0 + \phi)\chi$$

K STRETCHED STRINGS

$$= \begin{cases} K & \text{W-BOSON CREATION AT } t=-\infty \\ K & \text{W-BOSON ANNIHILATION AT } t=+\infty \end{cases}$$

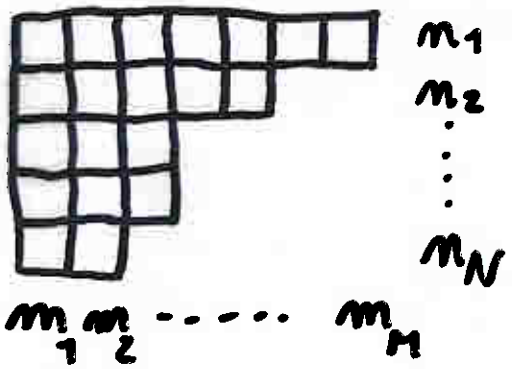
$$Z = e^{i S_{M=4}} \int [Dx][Dx^+]\, e^{iSx} \cdot$$

$$\cdot \frac{1}{K!} \sum_{i_1,\dots i_K} x_{\lambda_1}(\infty) \cdots x_{\lambda_K}(\infty)\, x^+_{\lambda_1}(+\infty) \cdots x^+_{\lambda_K}(+\infty)$$

$$= e^{i S_{M=4}}$$ 

$D3_K \longleftrightarrow$

$\frac{1}{2}$ BPS WILSON LOOP IN A GENERIC REPRESENTATION



IS ASSOCIATED TO

$$(D5_{m_1}, D5_{m_2}, \ldots, D5_{m_M})$$

OR

$$(D3_{m_1}, D3_{m_2}, \ldots D3_{m_N})$$