Overview and Perspectives in Gauge Theory on Compact Manifolds and AGT-like relations (II):

3d & 5d partition functions as q-CFT correlators

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CERN

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based on arXiv:1303.2626, with F. Nieri and S. Pasquetti and work in progress

Motivations

- Recently, many exact results for gauge theories on compact manifolds using localization techniques
  - useful to study holography, 3d-3d duality, holomorphic blocks (Sara's talk), ...

Most relevant for this talk:

- AGT correspondence: relates certain BPS observables in 4d and 2d gauge theories to Liouville/Toda CFT correlators
  [Alday-Gaiotto-Tachikawa],[Wyllard]
  - classification of line operators
  - VEV of BPS line operators and S-duality action on them
  - VEV of surface operators
  - VEV of domain walls

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- Today: focus on 3d and 5d gauge theory partition functions and relate them to correlators with underlying q-Virasoro symmetry

Localization for gauge theory on compact manifolds

SUSY theory on a compact manifold $\mathcal{M}$ of dimension d, with fields Ψ

Localization: $Z_{\mathcal{M}} = \int D\psi e^{-S[\Psi]} = \int D\Psi_0 e^{-S[\Psi_0]} Z_{1-loop}[\Psi_0]$

- Ψ_0: field configurations satisfying localizing (saddle point) equations
- a different localization schemes can produce different set of saddle points, $\tilde{\Psi}_0$

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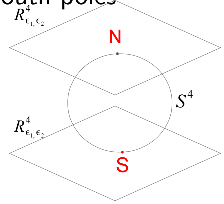
$\mathcal{N} = 2$ theories on S^4 [Pestun]

Ψ_0 given by:
- zero mode of a vector multiplet scalar a
- point-like instantons at North and South poles



$$Z_{S^4} = \int \prod_i da_i Z_{cl}(\vec{a},\tau) Z_{1-loop}(\vec{a},\vec{m};\epsilon_1,\epsilon_2)$$
$$\times \left| Z_{inst}(\vec{a},\vec{m},\tau;\epsilon_1,\epsilon_2) \right|^2$$

- $Z_{inst}(\vec{a},\vec{m},\tau;\epsilon_1,\epsilon_2)$ is the instanton partition function on $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$.

Liouville theory

Liouville CFT data

- $c_V = 1 + 6Q_0^2$ is the Virasoro central charge $(Q_0 = b_0 + 1/b_0)$
- $V_\alpha(z, \tilde{z})$ primary field with conformal dimension $\Delta_\alpha = \alpha(Q_0 - \alpha)$
  - $\alpha = \frac{Q_0}{2} + ip_\alpha$ with $p_\alpha \in \mathbb{R}$: non-degenerate representation
  - $\alpha^{(m,n)} = \frac{Q_0}{2} - \frac{m}{2b_0} - \frac{nb_0}{2}$ with $n, m \in \mathbb{N}$: degenerate representation
- $C(\alpha, \beta, \gamma)$ 3-point correlation function, DOZZ formula

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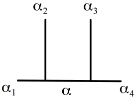
Any *n*-point function is decomposed in terms of 3-point functions

Basic example: 4-point function

$$\langle \alpha_4 | V_{\alpha_3}(1) V_{\alpha_2}(\zeta) | \alpha_1 \rangle = \int d\alpha$$



$$= \int \mathrm{d}\alpha\, C(\alpha_4^*, \alpha_3, \alpha) C(\alpha^*, \alpha_2, \alpha_1) |\zeta^{\Delta_\alpha - \Delta_{\alpha_2} - \Delta_{\alpha_1}} \mathcal{F}_{\alpha_4\alpha_3\alpha\alpha_2\alpha_1}(\zeta)|^2$$

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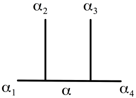

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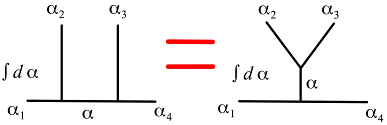
Consistent CFT data satisfy crossing symmetry

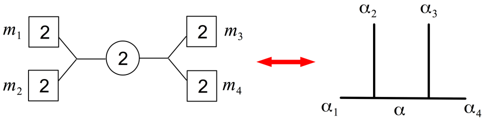


AGT correspondence

S^4 gauge theory BPS observables $\Longleftrightarrow$ Liouville/Toda CFT correlators

[Alday-Gaiotto-Tachikawa]

The prototypical AGT example: S^4 partition function of $SU(2)$ SCQCD ($N_f = 4$) is equivalent to a 4-point non-degenerate correlator



$$Z_{S^4}^{SCQCD} = \langle \alpha_4 | V_{\alpha_3}(1) V_{\alpha_2}(\zeta) | \alpha_1 \rangle$$

Dictionary:

$2d$ CFT	$4d$ gauge theory
conformal blocks $\mathcal{F}_{\alpha_4 \alpha_3 \sigma \alpha_2 \alpha_1}$	instanton contribution Z_{inst}
$3-$point functions $C(..)C(..)$	perturbative contribution $Z_{1-\text{loop}}$
external momenta α_i	masses m_i
internal momentum α	coulomb branch a

why AGT works

- Perturbative tests and direct proofs in special cases ($\mathcal{N} = 2^*$, pure $SU(N)$, $\epsilon_1 + \epsilon_2 = 0$ etc).

  recent mathemetical results [Maulik-Okounkov],[Schiffmann-Vasserot]

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S^4 partition functions and Liouville correlators constrained by the same symmetries $\Rightarrow$ they are solutions of the same bootstrap equations, therefore are equal!

AGT with surface operators

surface operators: codimension-2 operators defined by the path integral in the presence of prescribed singularities along the defect surface.
They can be defined also coupling the 4d gauge theory to a 2d field theory on the defect.[Gukov-Witten]

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Simple surface operators $\Leftrightarrow$ SQED with N chirals and N antichirals

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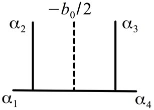
Simple surface operators$\Leftrightarrow$degenerate primaries $\left(L_{-2} + \frac{1}{b^2} L_{-1}^2\right) V_{-b/2} = 0$
[Alday-Gaiotto-Gukov-Tachikawa-Verlinde]

$\langle \text{surface operator} \rangle_{SCQCD} =$ 

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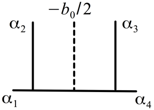
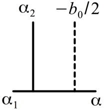
$\langle \text{surface operator} \rangle_{SCQCD} =$ 

- Decoupling 2d theory from 4d theory [Doroud-Gomis-Le Floch-Lee]



$= \langle V_{\alpha_4} V_{\alpha_3}(1) V_{-b/2}(z) V_{\alpha_1} \rangle = Z^{SQED}_{S_2}$

- Previous results: degenerate conformal blocks $\leftrightarrow$2d vortex counting [Dimofte-Gukov-Hollands],[Kozcaz-Pasquetti-Wyllard],[Bonelli-Tanzini-Zhao].

$\mathcal{N} = (2,2)$ theory on S^2

Two localization schemes $\rightarrow$ two representations of the partition function

[Benini-Cremonesi],[Doroud-Gomis-Le Floch-Lee]

SQED: $U(1)$ gauge group, N_f chirals m_i, N_f anti-chirals $\tilde{m}_j$, with FI ξ.

- Coulomb branch:

$$Z_{S^2} = \sum_{s \in Z} \int da \; Z_{cl}(a, s, \xi, \theta) \; Z_{1-loop}(a, \vec{m}, s)$$

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$$Z_{S^2} = \sum_{i}^{N_f} Z_{cl}^{(i)} Z_{1-loop}^{(i)} \left|\left| Z_V^{(i)}(\vec{m}; e^{2\pi i \tau}) \right|\right|^2$$

  - $Z_{cl}^{(i)} Z_{1-loop}^{(i)}$ are evaluated on Higgs vacua $a = a^{(i)}(\vec{m})$.
  - $Z_V^{(i)}$ is the vortex partition function on $\mathbb{R}^2_\epsilon$: [Shadchin]

$$Z_V^{(i)}(\vec{m}; e^{2\pi i \tau}) = \sum_k \prod_{j=1}^{N_f} \frac{(-im_i - i\tilde{m}_j)_k}{k!(1 + im_j - im_i)_k} e^{2\pi i \tau k} = {}_{N_f}F_{N_f - 1}^{(i)}(e^{2\pi i \tau})$$

  - where $\tau = \frac{\theta}{2\pi} + i\xi$, and the pairing: $\left|\left| f(a; z) \right|\right|^2 = f(a; z) f(a; \bar{z})$.

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Flop Symmetry: Z_{S^2} is invariant under : $\xi \leftrightarrow -\xi$ and $m_j \leftrightarrow -\tilde{m}_j$

The Bootstrap approach to Liouville

Conformal Bootstrap Approach: impose crossing symmetry to derive 3-point functions. Useful to consider degenerate representation of Virasoro. [Belavin-Polyakov-Zamolodchikov]

- 4-point function with a degenerate insertion [Teschner]

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{-b_0/2}(z,\tilde{z}) V_{\alpha_1}(0) \rangle \sim G(z,\tilde{z})$$

- $V_{-b_0/2}(z,\tilde{z})$ has a null state at level 2, leading to

$$D(a,b;c;z) G(z,\tilde{z}) = 0\,, \qquad D(a,b;c;\tilde{z}) G(z,\tilde{z}) = 0$$

where $D(a,b;c;z)$ is the hypergeometric differential operator

$$D(a,b;c;z) = z(1-z)\frac{\partial^2}{\partial z^2} + [c-(a+b+1)z]\frac{\partial}{\partial z} - ab$$

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$G(z,\tilde{z})$ is a bilinear combination of solutions defined around singular points $0, 1, \infty$

Around $z = 0$

$$I_1^{(s)}(z) = {}_2F_1(a, b; c; z), \qquad I_2^{(s)}(z) = z^{1-c}\, {}_2F_1(1 + a - c, 1 + b - c; 2 - c; z)$$

this is the s-channel

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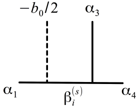
$$= \sum_i$$



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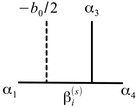

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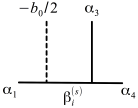

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where the internal states are $\beta_1^{(s)} = \alpha_1 - \frac{b_0}{2}$, $\beta_2^{(s)} = \alpha_1 + \frac{b_0}{2}$

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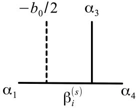
$$= \sum_i$$ 

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- pairing

$$\left|\left| f(a,b,c,z) \right|\right|^2 = f(a,b,c,z) f(a,b,c,\bar{z})$$

Around $z = \infty$

$$I_1^{(u)}(z) = z^{-a}\,{}_2F_1(a, 1+a-c; 1+a-b; z^{-1})$$
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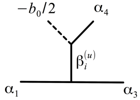

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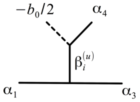
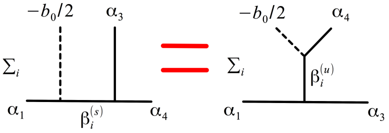


Conformal Bootstrap:



extending a set of solutions by analytic continuation $I_i^{(s)} = \sum_{j=1}^{2} M_{ij} I_j^{(u)}$, we obtain

$$\sum_{k,l=1}^{2} K_{kl}^{(s)} M_{ki} M_{lj} = K_{ij}^{(u)}$$

This set of equations determines the 3-point functions.

- off-diagonal elements

$$\frac{K_{22}^{(s)}}{K_{11}^{(s)}} = -\frac{M_{11} M_{12}}{M_{21} M_{22}}$$

- diagonal elements

$$\frac{K_{22}^{(u)}}{K_{22}^{(s)}} = (M_{22})^2 - \frac{M_{21} M_{12} M_{22}}{M_{11}} = \frac{M_{22}}{M_{11}} (\det M)$$

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The DOZZ formula solves both the equations

$$C(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{\Upsilon(2\alpha_T - Q_0)} \prod_{r=1}^{3} \frac{\Upsilon(2\alpha_r)}{\Upsilon(2\alpha_T - 2\alpha_r)}$$

where $2\alpha_T = \alpha_1 + \alpha_2 + \alpha_3$ and

$$\Upsilon(X) \propto \prod_{n_1, n_2 = 0}^{\infty} (X + n_1 b_0 + n_2 1/b_0)(-X + (n_1+1)b_0 + (n_2+1)1/b_0)$$

Conformal Bootstrap approach: 3-point function is derived exploiting symmetries, without using the Lagrangian.

Comments on the relation: $Z_{S_2}^{SQED} = \langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{-b/2}(z) V_{\alpha_1}(0) \rangle$

- vortices moduli space are slices of instantons moduli space. Indeed

  2d vortex partition functions $\Leftrightarrow$ degenerate conformal blocks

- flop transformations correspond to different channel decomposition of the correlator.

  flop symmetry invariance of Z_{S^2} $\Leftrightarrow$ correlator crossing symmetry

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We will now argue that a similar story holds in 3d. We start by reviewing 3d partition functions.

$\mathcal{N} = 2$ theory on S_b^3

$$S_b^3 : \quad b^2|z_1|^2 + \frac{1}{b^2}|z_2|^2 = 1$$

Coulomb branch localization scheme [Hama-Hosomichi-Lee].

SQED: $U(1)$ gauge group, N_f chirals m_j, N_f anti-chirals $\tilde{m}_k$, with FI ξ.

$$Z_S^{SQED} = \int dx \; G_{cl} \cdot G_{1-loop} = \int dx \; e^{2\pi i x \xi} \prod_{j,k}^{N_f} \frac{s_b(x + m_j + iQ/2)}{s_b(x + \tilde{m}_k - iQ/2)}$$

The 1-loop contribution of a chiral multiplet is:

$$s_b(x) = \prod_{m,n \in \mathbb{Z}_{\geq 0}} \frac{mb + nb^{-1} + \frac{Q}{2} - ix}{mb + nb^{-1} + \frac{Q}{2} + ix} , \qquad Q = b + 1/b \, .$$

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Flop Symmetry: Z_S^{SQED} is invariant under : $m_j \leftrightarrow -\tilde{m}_k$ and $\xi \leftrightarrow -\xi$
this signs flip exchanges phase I and phase II

Higgs-branch-like factorized form: [Pasquetti]

$$Z_S^{SQED} = \sum_i^{N_f} G_{cl}^{(i)} G_{1-loop}^{(i)} \left|\left| \mathcal{Z}_V^{(i)} \right|\right|_S^2 ,$$

- $G_{cl}^{(i)} G_{1-loop}^{(i)}$ are evaluated on Higgs vacua $x = -m_i$

$$G_{cl}^{(i)} = e^{-2\pi i \xi m_i}, \qquad G_{1-loop}^{(i)} = \prod_{j,k}^{N_f} \frac{s_b(m_j - m_i + iQ/2)}{s_b(\tilde{m}_k - m_i - iQ/2)} ,$$

- q-deformed 2d vortices:

$$\mathcal{Z}_V^{(i)} = \sum_n \prod_{j,k}^{N_f} \frac{(y_k x_i^{-1}; q)_n}{(q x_j x_i^{-1}; q)_n} z^n = {}_{N_f}\Phi_{N_f-1}^{(i)}(\vec{x}, \vec{y}; z) .$$

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- Vortices are glued with S-pairing:

$$\left|\left| f(x;q) \right|\right|_S^2 = f(x;q) f(\tilde{x};\tilde{q})$$

$$x = e^{2\pi X b} \Leftrightarrow \tilde{x} = e^{2\pi X/b}, \quad q = e^{2\pi i b^2} \Leftrightarrow \tilde{q} = e^{2\pi i/b^2}$$

where x variables are $x_i = e^{2\pi m_i b}$, $\quad y_i = e^{2\pi \tilde{m}_i b}$, $\quad z = e^{2\pi \xi b}$

$\mathcal{N} = 2$ theory on $S^2 \times S^1$

Computes the (generalized) super-conformal-index

[Imamura-Yokoyama],[Kapustin-Willet],[Dimofte-Gukov-Gaiotto].

SQED with fugacities:

$$
\begin{aligned}
&(\phi_i, r_i), \qquad i = 1, \cdots N_f, \qquad \text{flavor} \quad U(1)^{N_f}\,,\\
&(\xi_i, l_i), \qquad i = 1, \cdots N_f, \qquad \text{(anti)} - \text{flavor} \quad U(1)^{N_f}\,,\\
&(\omega, n), \qquad \text{topological} \quad U(1)\,,\\
&(t, s), \qquad \text{gauged} \quad U(1)\,.
\end{aligned}
$$

$$
Z_{id} = \sum_{s \in \mathbb{Z}} \int \frac{dt}{2\pi i t} t^n \omega^s \prod_{j=1}^{N_f} \chi(t\phi_j, s + r_j) \prod_{k=1}^{N_f} \chi(t^{-1}\xi_k^{-1}, -s - l_k)\,.
$$

The 1-loop contribution of a chiral multiplet is:

$$
\chi(\zeta, m) = (q^{1/2}\zeta^{-1})^{-m/2} \prod_{k=0}^{\infty} \frac{(1 - q^{k+1}\zeta^{-1}q^{-m/2})}{(1 - q^k \zeta q^{-m/2})}
$$

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Flop Symmetry:

Z_{id}^{SQED} is invariant under : $\omega \leftrightarrow \omega^{-1},\ n \leftrightarrow -n,\ \phi_j \leftrightarrow \xi_j^{-1},\ r_j \leftrightarrow -l_j$

exchanges phase I and phase II

Higgs-branch-like factorized form: [Beem-Dimofte-Pasquetti],[Hwang-Kim-Park]

$$Z_{id}^{SQED} = \sum_{i=1}^{N_f} G_{cl}^{(i)} G_{1loop}^{(i)} \left|\left| Z_V^i \right|\right|_{id}^2 ,$$

- $G_{cl}^{(i)} G_{1-loop}^{(i)}$ are evaluated on Higgs vacua $t = \phi_i^{-1}$, $s = -r_i$:

$$G_{cl}^{(i)} = \omega^{-r_i} (\phi_i^{-1})^n, \quad G_{1loop}^{(i)} = \prod_{j=1}^{N_f} \chi(\phi_j \phi_i^{-1}, r_j - r_i) \prod_{k=1}^{N_f} \chi(\phi_i \xi_k^{-1}, r_i - l_k) ,$$

- q-deformed 2d vortices:

$$\mathcal{Z}_V^{(i)} = \sum_n \prod_{j,k}^{N_f} \frac{(y_k x_i^{-1}; q)_n}{(q x_j x_i^{-1}; q)_n} z^n = {}_{N_f}\Phi_{N_f-1}^{(i)}(\vec{x}, \vec{y}; z) .$$

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$$\left|\left| f(x;q) \right|\right|_{id}^2 := f(x;q) f(\tilde{x};\tilde{q}) ,$$

$$x \Leftrightarrow \tilde{x} = \bar{x}, \quad q \Leftrightarrow \tilde{q} = q^{-1}$$

where x variables are $x_i = \phi_i q^{r_i/2}$, $y_i = \xi_i q^{l_i/2}$, $z = \omega q^{n/2}$

Flop symmetry

- On the Coulomb branch it's a trivial symmetry of the integrand
- On the Higgs branch it translates into highly non-trivial relations between partition function in the two phases (analytic continuation in $z \to z^{-1}$)

$$
\begin{aligned}
Z^{I}_{id} &= \sum_{i}^{N_f} G^{(i),I}_{cl}\, G^{(i),I}_{1loop} \left|\left|\mathcal{Z}^{(i),I}_{V}\right|\right|^2_{id} = \\
&= \sum_{i}^{N_f} G^{(i),II}_{cl}\, G^{(i),II}_{1loop} \left|\left|\mathcal{Z}^{(i),II}_{V}\right|\right|^2_{id} = Z^{II}_{id}\,,
\end{aligned}
$$

$$
\begin{aligned}
Z^{I}_{S} &= \sum_{i}^{N_f} G^{(i),I}_{cl}\, G^{(i),I}_{1loop} \left|\left|\mathcal{Z}^{(i),I}_{V}\right|\right|^2_{S} = \\
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\end{aligned}
$$

$\Rightarrow$Can these relations constrain the 1-loop part of the partition function?

3d partition functions and q-deformed CFT

3d partition functions, in the Higgs branch expression, have a structure similar to degenerate Liouville correlators

$$Z_{id,S} = \sum_{i=1}^{N_f} G_{cl}^{(i)} G_{1loop}^{(i)} \left|\left| Z_V^i \right|\right|^2_{\mathrm{id,S}}$$

- gauge theory flop symmetry $\Rightarrow$ crossing symmetry
- q-deformed hypers as conformal blocks $\Rightarrow$ q-deformation of Virasoro

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We will construct q-deformed degenerate correlators using the bootstrap approach. These correlators are equivalent to 3d partition functions.

q-deformed Virasoro algebra $\mathcal{V}ir_{q,t}$

$\mathcal{V}ir_{q,t}$ has two complex parameters q, t (useful to consider $p = \frac{q}{t}$) and an infinite set of generators T_n with $n \in \mathbb{Z}$

[Shiraishi, Kubo, Awata, Odake][Lukyanov-Pugai] [Frenkel-Reshetikhin][Jimbo-Miwa]

$$[T_n\,,\ T_m] = -\sum_{l=1}^{+\infty} f_l\,(T_{n-l}T_{m+l} - T_{m-l}T_{n+l}) - \frac{(1-q)(1-t^{-1})}{1-p}(p^n - p^{-n})\delta_{m+n,0}$$

where

$$f(z) = \sum_{l=0}^{+\infty} f_l z^l = \exp\left[\sum_{l=1}^{+\infty} \frac{1}{n}\frac{(1-q^n)(1-t^{-n})}{1+p^n} z^n\right]$$

- invariant under: $(q,t) \to (q^{-1}, t^{-1})$ and $(q,t) \to (t,q)$
- considering: $t = q^{-b_0^2}$ and $q \to 1$
  $\mathcal{V}ir_{q,t}$ reduces to the Virasoro with central charge $c_V = 1 + 6Q_0^2$

($T(z) = \sum_n T_n z^{-n}$ reduces to the Virasoro current $L(z) = \sum_n L_n z^{-n-2}$)

Representations of $\mathcal{V}ir_{q,t}$ can be constructed using Verma modules

- The highest weight state $|\lambda\rangle$ satisfies

$$T_0|\lambda\rangle = \lambda|\lambda\rangle, \qquad T_n|\lambda\rangle = 0 \quad \text{for} \quad n > 0$$

- the Verma module $\mathcal{M}(\lambda)$ is constructed acting on the highest weight state $|\lambda\rangle$ with the operators T_{-n} with $n > 0$

- Verma modules can include singular states. In particular, there is a level 2 singular vector for the following values of the parameter λ

$$\lambda_1 = p^{1/2}q^{1/2} + p^{-1/2}q^{-1/2}, \qquad \lambda_2 = p^{1/2}t^{-1/2} + p^{-1/2}t^{1/2}$$

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Chiral blocks with degenerate primaries satisfy difference equations

[Awata, Kubo, Morita, Odake, Shiraishi] [Awata, Yamada][Schiappa,Wyllard]

q-deformed CFT

q-deformed Bootstrap Approach: Consider degenerate chiral blocks, constrained to satisfy difference equations. Impose crossing symmetry to derive 3-point functions.

- 4-point function with a degenerate insertion

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(r) V_{\alpha_2}(z,\tilde{z}) V_{\alpha_1}(0)\rangle \sim G(z,\tilde{z})$$

- $V_{\alpha_2}(z,\tilde{z})$ has a null state at level 2, leading to

$$D(A,B;C;q;z)G(z,z) = 0\,, \qquad D(\tilde{A},\tilde{B};\tilde{C};\tilde{q};\tilde{z})G(z,\tilde{z}) = 0\,,$$

where $D(A,B;C;q;z)$ is the q-hypergeometric operator

$$D(A,B;C;q;z) = h_2 \frac{\partial_q^2}{\partial_q z^2} + h_1 \frac{\partial_q}{\partial_q z} + h_0$$

and $\frac{\partial_q}{\partial_q z}$ is the q-derivative. It acts on a function $f(z)$ as

$$\frac{\partial_q}{\partial_q z} f(z) = \frac{f(qz) - f(z)}{z(q-1)}$$

$G(z,\tilde{z})$ is a bilinear combination of solutions of the q-hypergeometric eq.

Around $z = 0$

$$I_1^{(s)} = {}_2\Phi_1(A, B; C; z)$$
$$I_2^{(s)} = \frac{\theta(q^2 C^{-1} z^{-1}; q)}{\theta(qC^{-1}; q)\theta(qz^{-1}; q)} \, {}_2\Phi_1(qAC^{-1}, qBC^{-1}; q^2 C^{-1}; z)$$

- Virasoro limit $q \to 1$: $\lim_{q\to 1} {}_2\Phi_1(q^a, q^b; q^c; q, z) = {}_2F_1(a, b; c; z)$ and the solutions become the s-channel basis discussed above

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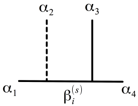
so in the s-channel

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(r) V_{\alpha_2}(z) V_{\alpha_1}(0)\rangle \sim \sum_{i,j=1}^{2} \tilde{I}_i^{(s)} K_{ij}^{(s)} I_j^{(s)}$$

$$= \sum_{i=1}^{2} K_{ii}^{(s)} \left|\left| I_i^{(s)} \right|\right|_*^2 = \sum_i$$



- generic pairing $\left|\left|(\ldots)\right|\right|_*^2$. For instance

$$\left|\left| f(A, B, C; z; q) \right|\right|_*^2 = f(A, B, C; z; q) f(\tilde{A}, \tilde{B}, \tilde{C}; \tilde{z}; \tilde{q})$$

Around $z = \infty$

$$I_1^{(u)} = \frac{\theta(qA^{-1}z^{-1};q)}{\theta(A^{-1};q)\theta(qz^{-1};q)}\ {}_2\Phi_1(A, qAC^{-1}; qAB^{-1}; q^2z^{-1})\,,$$

$$I_2^{(u)} = \frac{\theta(qB^{-1}z^{-1};q)}{\theta(B^{-1};q)\theta(qz^{-1};q)}\ {}_2\Phi_1(B, qBC^{-1}; qBA^{-1}; q^2z^{-1})$$

- in the $q \to 1$ limit we recover the undeformed u-channel basis

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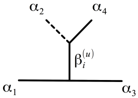
- in the $q \to 1$ limit we recover the undeformed u-channel basis

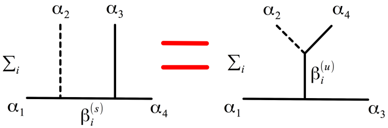
The correlation function in the u-channel is

$$\langle V_{\alpha_4}(\infty)V_{\alpha_3}(r)V_{\alpha_2}(z)V_{\alpha_1}(0)\rangle \sim \sum_{i,j=1}^{2} \tilde{I}_i^{(u)} K_{ij}^{(u)} I_j^{(s)}$$

$$= \sum_{i=1}^{2} K_{ii}^{(u)} \left|\left| I_i^{(u)} \right|\right|_*^2 = \sum_i$$



q-deformed Bootstrap:



$$K_{11}^{(s)}\left|\left|I_1^{(s)}\right|\right|_*^2 + K_{22}^{(s)}\left|\left|I_2^{(s)}\right|\right|_*^2 = K_{11}^{(u)}\left|\left|I_1^{(u)}\right|\right|_*^2 + K_{22}^{(u)}\left|\left|I_2^{(u)}\right|\right|_*^2$$

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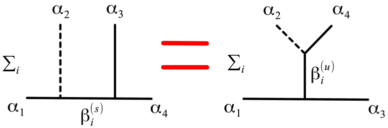

$$K_{11}^{(s)}\left|\left|I_1^{(s)}\right|\right|_*^2+K_{22}^{(s)}\left|\left|I_2^{(s)}\right|\right|_*^2=K_{11}^{(u)}\left|\left|I_1^{(u)}\right|\right|_*^2+K_{22}^{(u)}\left|\left|I_2^{(u)}\right|\right|_*^2$$

using analytical continuation $I_i^{(s)}=\sum_{j=1}^{2}M_{ij}I_j^{(u)}$ and $\tilde{I}_i^{(s)}=\sum_{j=1}^{2}\tilde{M}_{ij}\tilde{I}_j^{(u)}$ we obtain

$$\sum_{k,l=1}^{2}K_{kl}^{(s)}\tilde{M}_{ki}M_{lj}=K_{ij}^{(u)}$$

Bootstrap equation that determines the 3-point functions. In details

- off-diagonal elements $$\frac{K_{22}^{(s)}}{K_{11}^{(s)}}=-\frac{\tilde{M}_{11}M_{12}}{\tilde{M}_{21}M_{22}}$$
- diagonal elements $$\frac{K_{22}^{(u)}}{K_{22}^{(s)}}=\frac{M_{22}}{\tilde{M}_{11}}(\det\tilde{M})$$

solutions to these equations exist only for certain pairings!

id-pairing *q*-CFT

Consider the case where the blocks are glued as

$$\left|\left|f(x;q)\right|\right|^2_{id} = f(x;q)f(\tilde{x};\tilde{q})\,.$$

- with: $$x = e^{\beta X}\,, \quad \tilde{x} = e^{-\beta X}\,, \quad \tilde{q} = q^{-1}$$
- in particular, $q = e^{\beta/b_0}$ and the x variables are

$$A = e^{\beta(\alpha_1+\alpha_3+\alpha_4-\frac{b_0}{2}-Q_0)}\,, \quad B = e^{\beta(\alpha_1+\alpha_3-\alpha_4-\frac{b_0}{2})}\,, \quad C = e^{\beta(2\alpha_1-b_0)}$$

id-pairing q-CFT

Consider the case where the blocks are glued as

$$\left|\left|f(x;q)\right|\right|_{id}^2 = f(x;q)f(\tilde{x};\tilde{q})\,.$$

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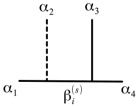
The bootstrap equations are solved by

$$C_{id}(\alpha_3,\alpha_2,\alpha_1) = \frac{1}{\Upsilon^\beta(2\alpha_T - Q_0)}\prod_{i=1}^{3}\frac{\Upsilon^\beta(2\alpha_i)}{\Upsilon^\beta(2\alpha_T - 2\alpha_i)}$$

- where we defined: $\Upsilon^\beta(X) \propto \prod_{k=-\infty}^{+\infty}\Upsilon\left(X + i\frac{2\pi}{\beta}k\right)$

$S^1 \times S^2$ superconformal index for SQED is equivalent to an *id*-pairing q-CFT degenerate correlator ($\alpha_2 = -b_0/2$)

$$Z_{id} = \sum_{i=1}^{N_f} G_{cl}^{(i),I}\, G_{1loop}^{(i),I} \left|\left| Z_V^{(i),I} \right|\right|^2_{\mathrm{id}} \sim \sum_{i=1}^{2} K_{ii}^{(s)} \left|\left| I_i^{(s)} \right|\right|^2_{\mathrm{id}} = \sum_i$$



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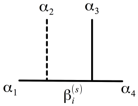

dictionary: writing the flavor fugacities as $\phi_i = e^{i\beta\,\Phi_i}$, $\xi_i = e^{i\beta\,\Xi_i}$

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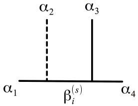

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where $\beta =$ circumference of S^1

- gauge theory flop symmetry $\Leftrightarrow$ q-CFT crossing symmetry
- $\beta \to 0$ limit
  - CFT: $\mathcal{V}ir_{q,t} \to$ Virasoro, we recover Liouville theory results
  - gauge: $S^1 \times S^2$ partition function reduce to S^2 partition function

  Consistent: S^2 partition functions match degenerate Liouville correlators [Doroud-Gomis-Le Floch-Lee]

S-pairing q-CFT

Now, consider the case where the blocks are glued as

$$\left|\left|f(x;q)\right|\right|_S^2 = f(x;q)f(\tilde{x};\tilde{q})\,.$$

- with:

$$x = e^{2\pi i X/\omega_2}\,, \quad \tilde{x} = e^{2\pi i X/\omega_1}, \quad q = e^{2\pi i \frac{\omega_1}{\omega_2}}\,, \quad \tilde{q} = e^{2\pi i \frac{\omega_2}{\omega_1}}$$

- and the x variables are

$$A = e^{2\pi i(\alpha_1+\alpha_3+\alpha_4-\frac{\omega_3}{2}-E)/\omega_2}\,, \quad B = e^{2\pi i(\alpha_1+\alpha_3-\alpha_4-\frac{\omega_3}{2})/\omega_2}\,,$$
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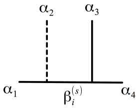
$$C_{\mathrm{S}}(\alpha_3, \alpha_2, \alpha_1) = \frac{1}{S_3(2\alpha_T - E)} \prod_{i=1}^{3} \frac{S_3(2\alpha_i)}{S_3(2\alpha_T - 2\alpha_i)}$$

where $S_3(X)$ is the triple sine function

$$S_3(X) \propto \prod_{n_1, n_2, n_3 = 0}^{+\infty} (\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 + X)(\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 + E - X)$$

S^3 partition function for SQED is equivalent to an S-pairing q-CFT degenerate correlator ($\alpha_2 = -\omega_3/2$)

$$Z_{\rm S} = \sum_{i=1}^{N_f} G_{cl}^{(i),I}\, G_{1loop}^{(i),I} \left|\left| Z_V^{(i),I} \right|\right|^2_{\rm S} \sim \sum_{i=1}^{2} K_{ii}^{(s)} \left|\left| I_i^{(s)} \right|\right|^2_{S} = \sum_{i}$$



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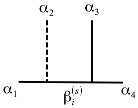

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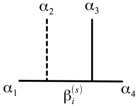

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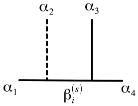
- gauge theory flop symmetry $\Leftrightarrow$ q-CFT crossing symmetry
- in fact there are 3 possibilities:

$$\alpha_2 = -\omega_k/2, \qquad q = e^{2\pi i \frac{\omega_i}{\omega_j}}\,, \qquad \tilde q = e^{2\pi i \frac{\omega_j}{\omega_i}}\,, \quad i \neq j \neq k = 1, 2, 3\,.$$

so far:

3d gauge theory partition functions $\Leftrightarrow$ q-CFT degenerate correlators

$$Z_{S,id} = \sum_{i=1}^{N_f} G_{cl}^{(i),I} G_{1loop}^{(i),I} \left|\left| Z_V^{(i),I} \right|\right|_{S,id}^2 \sim \sum_{i=1}^{2} K_{ii}^{(s)} \left|\left| I_i^{(s)} \right|\right|_{S,id}^2 = \sum_i$$



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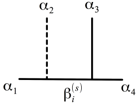

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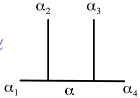
Let's now consider non-degenerate correlators

$$\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle_{\mathrm{S,id}} = \int d\alpha$$



$$= \int d\alpha\; C_{S,id}\; C_{S,id}\; (\mathrm{Conf.Blocks})$$

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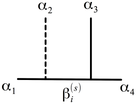

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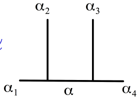

$$= \int d\alpha\, C_{S,id}\, C_{S,id}\, (\mathrm{Conf.Blocks})$$

in analogy with the AGT case, it is natural to expect that

5d gauge theory partition functions $\Leftrightarrow$ q-CFT non-degenerate correlators

$\mathcal{N} = 1$ theory on $S^4 \times S^1$

Computes 5d super-conformal index.

Coulomb branch localization yields: [Kim-Kim-Lee],[Terashima],[Iqbal,Vafa]

$$Z_{S^4 \times S^1} = \int d\vec{\sigma}\ \mathcal{Z}_{\text{1-loop}}(\vec{\sigma}, \vec{m}) \left| Z^{5d}_{inst}(\vec{\sigma}, \vec{m}, z; q, t) \right|^2$$

- $|Z^{5d}_{inst}|^2$ is the contribution of point-like instantons at N and S poles
- 1-loop contribution
  - vector multiplet:

$$\mathcal{Z}^{\text{vect}}_{\text{1-loop}}(\sigma) = \prod_{\alpha > 0} \Upsilon^{\beta}\left(i\alpha(\sigma)\right) \Upsilon^{\beta}\left(-i\alpha(\sigma)\right)$$

  - hypermultiplet of mass m in a representation R:

$$\mathcal{Z}^{\text{hyper}}_{\text{1-loop}}(\sigma, m, R) = \prod_{\rho \in R} \Upsilon^{\beta}\left(i(\rho(\sigma) + m) + \frac{Q_0}{2}\right)^{-1}$$

β=circumference of S^1

Conjecture: $Z_{S^4 \times S^1}$ correspond to non-degenerate correlators with $\mathcal{V}ir_{qt} \otimes \mathcal{V}ir_{qt}$ symmetry and *id*-pairing 3-point function.

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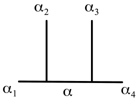
Example:

$$Z^{SQCD}_{S^4\times_q S^1}=\langle V_{\alpha_1}V_{\alpha_2}V_{\alpha_3}V_{\alpha_4}\rangle_{id}=\int d\alpha$$



- $5d$ instantons vs $\mathcal{V}ir_{qt}$ non-degenerate conformal blocks: [Awata-Yamada],[Mironov-Morozov-Shakirov-Smirnov]

$$Z^{5d,SCQCD}_{inst}=\mathcal{F}^{qt}_{\alpha_1\alpha_2\alpha\alpha_3\alpha_4}$$

- 1-loop vs 3-point function:

$$\mathcal{Z}^{\text{vect}}_{\text{1-loop}}(\sigma)\prod_{i=1}^{4}\mathcal{Z}^{\text{hyper}}_{\text{1-loop}}(\sigma,m_i,F)\quad=\quad C_{id}(\alpha_1,\alpha_2,\alpha)C_{id}(Q_0-\alpha,\alpha_3,\alpha_4)$$

with dictionary:

$\alpha=i\sigma+\frac{Q_0}{2}\,,\ \alpha_1\pm\alpha_2=im_{1,2}+Q_0\,,\quad \alpha_3\pm\alpha_4=im_{3,4}+Q_0$

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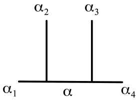

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$S^2 \times S^1$ is a codimension 2 defect $\rightarrow$ degenerate *id*-correlator

$\mathcal{N} = 1$ theory on S^5

$$\omega_1^2|z_1|^2 + \omega_2^2|z_2|^2 + \omega_3^2|z_3|^2 = 1$$

Coulomb branch localisation yields: [Kallen-Zabzine], [Hosomichi-Seong-Terashima],[Imamura],[Kim-Kim-Kim],[Lockart-Vafa],

$$Z_{S^5} = \int d\vec{\sigma}\; Z_{cl}(\vec{\sigma},\tau;\vec{\omega})\; Z_{1loop}(\vec{\sigma},\vec{m};\vec{\omega})$$
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- Instantons comes with equivariant parameters

$$(q,t):\quad \left(e^{2\pi i\frac{\omega_2}{\omega_1}}, e^{-2\pi i\frac{\omega_3}{\omega_1}}\right)_I,\ \left(e^{2\pi i\frac{\omega_1}{\omega_2}}, e^{-2\pi i\frac{\omega_3}{\omega_2}}\right)_{II},\ \left(e^{2\pi i\frac{\omega_1}{\omega_3}}, e^{-2\pi i\frac{\omega_2}{\omega_3}}\right)_{III}$$

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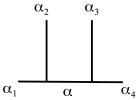

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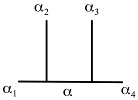

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- 1loop vs 3-point function:

$$\mathcal{Z}^{\text{vect}}_{\text{1loop}}(\sigma) \prod_{i=1}^{4} \mathcal{Z}^{\text{hyper}}_{\text{1-loop}}(\sigma, m_i, F) \quad = \quad C_S(\alpha_1, \alpha_2, \alpha) C_S(E-\alpha, \alpha_3, \alpha_4)$$

with dictionary: $\alpha = i\sigma + \frac{E}{2}\,, \quad \alpha_1 + \alpha_2 = im_1 + E\,, \quad \alpha_1 - \alpha_2 = im_2$ $\alpha_3 + \alpha_4 = im_3 + E\,, \quad \alpha_3 - \alpha_4 = im_4\,.$

S^3_b is a codimension 2 defect $\rightarrow$ degenerate S-correlator

Degeneration check: [Nieri, Pasquetti, Passerini, to appear]

For $\alpha_2 \to -\omega_3/2$ we have $\alpha \to \alpha^{\pm} = \alpha_1 \pm \omega_3/2$ and $Z^{SQCD}_{S^5} \to Z^{SQED}_{S^3_b}$

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In details

$$\int d\alpha \; Z_{1loop} \to \sum_{i=1}^{2} G^{(i)}_{1loop}\,, \qquad Z_{cl}\big|_{\alpha^{\pm}} \to G^{(1,2)}_{cl}\,,$$

$$Z^{5d,I}_{inst} = \sum_{Y_1,Y_2} (\cdots) \to \sum_{0,1^n} (\cdots) = \mathcal{Z}^{(1,2)}_V\,, \qquad Z^{5d,II}_{inst} = \sum_{W_1,W_2} (\cdots) \to \sum_{0,n} (\cdots) = \tilde{\mathcal{Z}}^{(1,2)}_V\,,$$

$$Z^{5d,III}_{inst} = \sum_{X_1,X_2} (\cdots) \to \sum_{0,0} (\cdots) = 1$$

so that

$$Z^{5d,I}_{inst} Z^{5d,II}_{inst} Z^{5d,III}_{inst} \to \left|\left|\mathcal{Z}^{(1,2)}_V\right|\right|^2_S$$

and similarly for permutations of $\omega_1, \omega_2, \omega_3$.

Degenerate correlators/$\mathcal{Z}^{SQED}_S$ are cross-symmetry/flop invariant.
$\to$Hints of cross-symmetry/S-duality invariance for S^5 theory

Conclusions and Outlook

- some evidence for a q-CFT-like structure of 5d and 3d partition functions
- use gauge theory to study q-CFT
  - consider other pairings to construct correlators
  - test of crossing symmetry in the non-degenerate case
- use q-CFT to study gauge theory
  - construct Verlinde loop operators and study their gauge theory meaning