

3d&5d partition functions as q-CFT correlators

Filippo Passerini

CERN

CERN, Geneva, May 14, 2013

based on arXiv:1303.2626, with F. Nieri and S. Pasquetti

Motivations

- ▶ Recently, many **exact results** for gauge theories on compact manifolds using **localization techniques**
 - ▶ useful to study holography
 - ▶ explicit connections between gauge theories and 2D CFT
- ▶ **AGT correspondence**: relates certain BPS observables in 4d and 2d gauge theories to Liouville/Toda CFT correlators
[Alday-Gaiotto-Tachikawa]
 - ▶ classification of line operators
 - ▶ VEV of BPS line operators and S-duality action on them
 - ▶ VEV of surface operators
 - ▶ VEV of domain walls

Motivations

- ▶ Recently, many **exact results** for gauge theories on compact manifolds using **localization techniques**
 - ▶ useful to study holography
 - ▶ explicit connections between gauge theories and 2D CFT
- ▶ **AGT correspondence**: relates certain BPS observables in 4d and 2d gauge theories to Liouville/Toda CFT correlators
[Alday-Gaiotto-Tachikawa]
 - ▶ classification of line operators
 - ▶ VEV of BPS line operators and S-duality action on them
 - ▶ VEV of surface operators
 - ▶ VEV of domain walls
- ▶ **In this talk**: focus on 3d and 5d gauge theory partition functions and relate them to CFT-correlators with underlying q Virasoro symmetry

Localization for gauge theory on compact manifolds

SUSY theory on a compact manifold \mathcal{M} of dimension d , with fields Ψ

Localization: $Z_{\mathcal{M}} = \int D\psi e^{-S[\psi]} = \int D\Psi_0 e^{-S[\Psi_0]} Z_{1-loop}[\Psi_0]$

- ▶ Ψ_0 : field configurations satisfying localizing (saddle point) equations
- ▶ with a clever localization scheme, Ψ_0 is a **finite dimensional set**
- ▶ $Z_{1-loop}[\Psi_0]$ is due to the quadratic fluctuations around Ψ_0

Localization for gauge theory on compact manifolds

SUSY theory on a compact manifold \mathcal{M} of dimension d , with fields Ψ

Localization: $Z_{\mathcal{M}} = \int D\psi e^{-S[\psi]} = \int D\Psi_0 e^{-S[\Psi_0]} Z_{1-loop}[\Psi_0]$

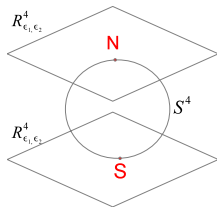
- ▶ Ψ_0 : field configurations satisfying localizing (saddle point) equations
- ▶ with a clever localization scheme, Ψ_0 is a **finite dimensional set**
- ▶ $Z_{1-loop}[\Psi_0]$ is due to the quadratic fluctuations around Ψ_0

$\mathcal{N} = 2$ theories on S^4 [Pestun]

Ψ_0 given by:

- **zero mode** of a vector multiplet scalar a
- **point-like instantons** at North and South poles

$$Z_{S^4} = \int \prod_i da_i Z_{cl}(\vec{a}, \tau) Z_{1-loop}(\vec{a}, \vec{m}; \epsilon_1, \epsilon_2) \times \left| Z_{inst}(\vec{a}, \vec{m}, \tau; \epsilon_1, \epsilon_2) \right|^2$$



- ▶ $Z_{inst}(\vec{a}, \vec{m}, \tau; \epsilon_1, \epsilon_2)$ is the instanton partition function on $\mathbb{R}^4_{\epsilon_1, \epsilon_2}$.

Liouville theory

Liouville CFT data

- ▶ $c_V = 1 + 6Q_0^2$ is the Virasoro central charge ($Q_0 = b_0 + 1/b_0$)
- ▶ $V_\alpha(z, \bar{z})$ primary field with conformal dimension $\Delta_\alpha = \alpha(Q_0 - \alpha)$
 - ▶ $\alpha = \frac{Q_0}{2} + ip_\alpha$ with $p_\alpha \in \mathbb{R}$: non-degenerate representation
 - ▶ $\alpha^{(m,n)} = \frac{Q_0}{2} - \frac{m}{2b_0} - \frac{nb_0}{2}$ with $n, m \in \mathbb{N}$: degenerate representation
- ▶ $C(\alpha, \beta, \gamma)$ 3-point correlation function, DOZZ formula

Liouville theory

Liouville CFT data

- ▶ $c_V = 1 + 6Q_0^2$ is the Virasoro central charge ($Q_0 = b_0 + 1/b_0$)
- ▶ $V_\alpha(z, \bar{z})$ primary field with conformal dimension $\Delta_\alpha = \alpha(Q_0 - \alpha)$
 - ▶ $\alpha = \frac{Q_0}{2} + ip_\alpha$ with $p_\alpha \in \mathbb{R}$: non-degenerate representation
 - ▶ $\alpha^{(m,n)} = \frac{Q_0}{2} - \frac{m}{2b_0} - \frac{nb_0}{2}$ with $n, m \in \mathbb{N}$: degenerate representation
- ▶ $C(\alpha, \beta, \gamma)$ 3-point correlation function, DOZZ formula

Any n -point function is decomposed in terms of 3-point functions

Basic example: 4-point function

$$\begin{aligned} \langle \alpha_4 | V_{\alpha_3}(1) V_{\alpha_2}(\zeta) | \alpha_1 \rangle &= \int d\alpha \begin{array}{c} \alpha_2 \quad \alpha_3 \\ | \quad | \\ \hline \alpha_1 \quad \alpha \quad \alpha_4 \end{array} \\ &= \int d\alpha C(\alpha_4^*, \alpha_3, \alpha) C(\alpha^*, \alpha_2, \alpha_1) |\zeta^{\Delta_\alpha - \Delta_{\alpha_2} - \Delta_{\alpha_1}} \mathcal{F}_{\alpha_4 \alpha_3 \alpha \alpha_2 \alpha_1}(\zeta)|^2 \end{aligned}$$

Liouville theory

Liouville CFT data

- ▶ $c_V = 1 + 6Q_0^2$ is the Virasoro central charge ($Q_0 = b_0 + 1/b_0$)
- ▶ $V_\alpha(z, \bar{z})$ primary field with conformal dimension $\Delta_\alpha = \alpha(Q_0 - \alpha)$
 - ▶ $\alpha = \frac{Q_0}{2} + ip_\alpha$ with $p_\alpha \in \mathbb{R}$: non-degenerate representation
 - ▶ $\alpha^{(m,n)} = \frac{Q_0}{2} - \frac{m}{2b_0} - \frac{nb_0}{2}$ with $n, m \in \mathbb{N}$: degenerate representation
- ▶ $C(\alpha, \beta, \gamma)$ 3-point correlation function, DOZZ formula

Any n -point function is decomposed in terms of 3-point functions

Basic example: 4-point function

$$\begin{aligned} \langle \alpha_4 | V_{\alpha_3}(1) V_{\alpha_2}(\zeta) | \alpha_1 \rangle &= \int d\alpha \begin{array}{c} \alpha_2 \quad \alpha_3 \\ | \quad | \\ \hline \alpha_1 \quad \alpha \quad \alpha_4 \end{array} \\ &= \int d\alpha C(\alpha_4^*, \alpha_3, \alpha) C(\alpha^*, \alpha_2, \alpha_1) |\zeta^{\Delta_\alpha - \Delta_{\alpha_2} - \Delta_{\alpha_1}} \mathcal{F}_{\alpha_4 \alpha_3 \alpha \alpha_2 \alpha_1}(\zeta)|^2 \end{aligned}$$

Consistent CFT data satisfy crossing symmetry

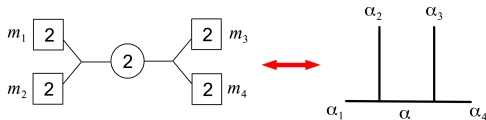
$$\int d\alpha \begin{array}{c} \alpha_2 \quad \alpha_3 \\ | \quad | \\ \hline \alpha_1 \quad \alpha \quad \alpha_4 \end{array} = \int d\alpha \begin{array}{c} \alpha_2 \quad \alpha_3 \\ \diagdown \quad / \\ \hline \alpha \\ \diagup \quad \diagdown \\ \alpha_1 \quad \alpha_4 \end{array}$$

AGT correspondence

S^4 gauge theory BPS observables \iff Liouville/Toda CFT correlators

[Alday-Gaiotto-Tachikawa]

The prototypical AGT example: S^4 partition function of $SU(2)$ SCQCD ($N_f = 4$) is equivalent to a 4-point non-degenerate correlator



$$Z_{S^4}^{SCQCD} = \langle \alpha_4 | V_{\alpha_3}(1) V_{\alpha_2}(\zeta) | \alpha_1 \rangle$$

Dictionary:

$2d$ CFT	$4d$ gauge theory
conformal blocks $\mathcal{F}_{\alpha_4 \alpha_3 \sigma \alpha_2 \alpha_1}$	instanton contribution Z_{inst}
3 – point functions $C(\cdot)C(\cdot)$	perturbative contribution $Z_{1\text{-loop}}$
external momenta α_i	masses m_i
internal momentum α	coulomb branch a

why AGT works

- ▶ Perturbative tests and direct proofs in special cases

Mathematical results [Maulik-Okounkov],[Schiffmann-Vasserot]

instanton partition functions \Leftrightarrow conformal blocks

- ▶ generalized $\mathcal{N} = 2$ S-dualities (Gaiotto surface pants-decomposition) correspond to different channel decomposition of the correlator.

S-duality invariance of $Z_{S^4} \Leftrightarrow$ crossing symmetry

why AGT works

- ▶ Perturbative tests and direct proofs in special cases

Mathematical results [Maulik-Okounkov],[Schiffmann-Vasserot]

instanton partition functions \Leftrightarrow conformal blocks

- ▶ generalized $\mathcal{N} = 2$ S-dualities (Gaiotto surface pants-decomposition) correspond to different channel decomposition of the correlator.

S-duality invariance of $Z_{S^4} \Leftrightarrow$ crossing symmetry

S^4 partition functions and Liouville correlators constrained by the same symmetries \Rightarrow they are solutions of the same bootstrap equations, therefore are equal!

AGT with surface operators

surface operators: codimension-2 operators defined by the path integral in the presence of prescribed singularities along the defect surface.

They can be defined also coupling the 4d gauge theory to a 2d field theory on the defect.

AGT with surface operators

surface operators: codimension-2 operators defined by the path integral in the presence of prescribed singularities along the defect surface. They can be defined also coupling the 4d gauge theory to a 2d field theory on the defect.

$SU(N)$ SCQCD:

Simple surface operators \Leftrightarrow SQED with N chirals and N antichirals

AGT with surface operators

surface operators: codimension-2 operators defined by the path integral in the presence of prescribed singularities along the defect surface. They can be defined also coupling the 4d gauge theory to a 2d field theory on the defect.

$SU(N)$ SCQCD:

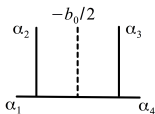
Simple surface operators \Leftrightarrow SQED with N chirals and N antichirals

AGT prescription:

Simple surface operators \Leftrightarrow degenerate primaries $(L_{-2} + \frac{1}{b^2} L_{-1}^2) V_{-b/2} = 0$

[Alday-Gaiotto-Gukov-Tachikawa-Verlinde]

$\langle \text{surface operator} \rangle_{SCQCD} =$



- Several results: degenerate conformal blocks \Leftrightarrow vortex counting
[Dimofte-Gukov-Hollands],[Kozcaz-Pasquetti-Wyllard],[Bonelli-Tanzini-Zhao].

AGT with surface operators

surface operators: codimension-2 operators defined by the path integral in the presence of prescribed singularities along the defect surface. They can be defined also coupling the 4d gauge theory to a 2d field theory on the defect.

$SU(N)$ SCQCD:

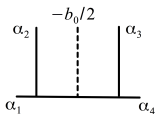
Simple surface operators \Leftrightarrow SQED with N chirals and N antichirals

AGT prescription:

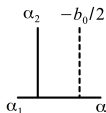
Simple surface operators \Leftrightarrow degenerate primaries $(L_{-2} + \frac{1}{b^2} L_{-1}^2) V_{-b/2} = 0$

[Alday-Gaiotto-Gukov-Tachikawa-Verlinde]

$\langle \text{surface operator} \rangle_{SCQCD} =$



- ▶ Several results: degenerate conformal blocks \leftrightarrow vortex counting [Dimofte-Gukov-Hollands],[Kozcaz-Pasquetti-Wyllard],[Bonelli-Tanzini-Zhao].
- ▶ Decoupling 2d theory from 4d theory [Doroud-Gomis-Le Floch-Lee]



$$= \langle V_{\alpha_4} V_{\alpha_3}(1) V_{-b/2}(z) V_{\alpha_1} \rangle = Z_{S_2}^{SQED}$$

$\mathcal{N} = (2, 2)$ theory on S^2

Two localization schemes \rightarrow two representations of the partition function

[Benini-Cremonesi],[Doroud-Gomis-Le Floch-Lee]

SQED: $U(1)$ gauge group, N_f chirals m_i , N_f anti-chirals \tilde{m}_k , with FI ξ .

► Coulomb branch:

$$Z_{S^2} = \sum_{s \in Z} \int da Z_{cl}(a, s, \xi, \theta) Z_{1-loop}(a, \vec{m}, s)$$

$\mathcal{N} = (2, 2)$ theory on S^2

Two localization schemes \rightarrow two representations of the partition function
[Benini-Cremonesi],[Doroud-Gomis-Le Floch-Lee]

SQED: $U(1)$ gauge group, N_f chirals m_i , N_f anti-chirals \tilde{m}_k , with FI ξ .

- ▶ Coulomb branch:

$$Z_{S^2} = \sum_{s \in Z} \int da Z_{cl}(a, s, \xi, \theta) Z_{1-loop}(a, \vec{m}, s)$$

- ▶ Higgs branch

$$Z_{S^2} = \sum_i^{N_f} Z_{cl}^{(i)} Z_{1-loop}^{(i)} \left\| Z_V^{(i)}(\vec{m}; e^{2\pi i \tau}) \right\|^2$$

- ▶ $Z_{cl}^{(i)} Z_{1-loop}^{(i)}$ are evaluated on Higgs vacua $a = a^{(i)}(\vec{m})$.
- ▶ $Z_V^{(i)}$ is the **vortex partition function** on \mathbb{R}_ϵ^2 : [Shadchin]

$$Z_V^{(i)}(\vec{m}; e^{2\pi i \tau}) = \sum_k \prod_{j=1}^{N_f} \frac{(-im_j - i\tilde{m}_j)_k}{k!(1 + im_j - im_j)_k} e^{2\pi i \tau k} = N_f F_{N_f-1}^{(i)}(e^{2\pi i \tau})$$

- ▶ where $\tau = \frac{\theta}{2\pi} + i\xi$, and the **pairing**: $\left\| f(a; z) \right\|^2 = f(a; z)f(a; \bar{z})$.

The Bootstrap approach to Liouville

Conformal Bootstrap Approach: impose crossing symmetry to derive 3-point functions. Useful to consider degenerate representation of Virasoro. [Belavin-Polyakov-Zamolodchikov]

- ▶ 4-point function with a **degenerate insertion** [Teschner]

$$\langle V_{\alpha_4}(\infty)V_{\alpha_3}(1)V_{-b_0/2}(z, \tilde{z})V_{\alpha_1}(0) \rangle \sim G(z, \tilde{z})$$

- ▶ $V_{-b_0/2}(z, \tilde{z})$ has a null state at level 2, leading to

$$D(a, b; c; z)G(z, \tilde{z}) = 0, \quad D(a, b; c; \tilde{z})G(z, \tilde{z}) = 0$$

where $D(a, b; c; z)$ is the **hypergeometric differential operator**

$$D(a, b; c; z) = z(1-z)\frac{\partial^2}{\partial z^2} + [c - (a+b+1)z]\frac{\partial}{\partial z} - ab$$

The Bootstrap approach to Liouville

Conformal Bootstrap Approach: impose crossing symmetry to derive 3-point functions. Useful to consider degenerate representation of Virasoro. [Belavin-Polyakov-Zamolodchikov]

- ▶ 4-point function with a **degenerate insertion** [Teschner]

$$\langle V_{\alpha_4}(\infty)V_{\alpha_3}(1)V_{-b_0/2}(z, \tilde{z})V_{\alpha_1}(0) \rangle \sim G(z, \tilde{z})$$

- ▶ $V_{-b_0/2}(z, \tilde{z})$ has a null state at level 2, leading to

$$D(a, b; c; z)G(z, \tilde{z}) = 0, \quad D(a, b; c; \tilde{z})G(z, \tilde{z}) = 0$$

where $D(a, b; c; z)$ is the **hypergeometric differential operator**

$$D(a, b; c; z) = z(1-z)\frac{\partial^2}{\partial z^2} + [c - (a+b+1)z]\frac{\partial}{\partial z} - ab$$

$G(z, \tilde{z})$ is a **bilinear combination of solutions** defined around singular points $0, 1, \infty$

Around $z = 0$

$$I_1^{(s)}(z) = {}_2F_1(a, b; c; z), \quad I_2^{(s)}(z) = z^{1-c} {}_2F_1(1+a-c, 1+b-c; 2-c; z)$$

this is the **s-channel**

Around $z = 0$

$$I_1^{(s)}(z) = {}_2F_1(a, b; c; z), \quad I_2^{(s)}(z) = z^{1-c} {}_2F_1(1+a-c, 1+b-c; 2-c; z)$$

this is the **s-channel**

$$\begin{aligned} \langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{\alpha_2}(z) V_{\alpha_1}(0) \rangle &\sim \sum_{i,j=1}^2 I_i^{(s)}(\bar{z}) K_{ij}^{(s)} I_j^{(s)}(z) = \sum_i K_{ii}^{(s)} \left\| I_i^{(s)}(z) \right\|^2 \\ &= \sum_i \begin{array}{c} \text{---} -b_0/2 \quad \alpha_3 \\ | \\ \alpha_1 \quad \beta_i^{(s)} \quad \alpha_4 \end{array} \end{aligned}$$

Around $z = 0$

$$I_1^{(s)}(z) = {}_2F_1(a, b; c; z), \quad I_2^{(s)}(z) = z^{1-c} {}_2F_1(1+a-c, 1+b-c; 2-c; z)$$

this is the **s-channel**

$$\begin{aligned} \langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{\alpha_2}(z) V_{\alpha_1}(0) \rangle &\sim \sum_{i,j=1}^2 I_i^{(s)}(\bar{z}) K_{ij}^{(s)} I_j^{(s)}(z) = \sum_i K_{ii}^{(s)} \left\| I_i^{(s)}(z) \right\|^2 \\ &= \sum_i \begin{array}{c} \text{---} \\ | \\ \alpha_1 \quad \beta_i^{(s)} \quad \alpha_4 \end{array} \end{aligned}$$

- ▶ diagonal monodromy around $z = 0 \rightarrow K_{12}^{(s)} = K_{21}^{(s)} = 0$

Around $z = 0$

$$I_1^{(s)}(z) = {}_2F_1(a, b; c; z), \quad I_2^{(s)}(z) = z^{1-c} {}_2F_1(1+a-c, 1+b-c; 2-c; z)$$

this is the **s-channel**

$$\begin{aligned} \langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{\alpha_2}(z) V_{\alpha_1}(0) \rangle &\sim \sum_{i,j=1}^2 I_i^{(s)}(\bar{z}) K_{ij}^{(s)} I_j^{(s)}(z) = \sum_i K_{ii}^{(s)} \left\| I_i^{(s)}(z) \right\|^2 \\ &= \sum_i \begin{array}{c} \alpha_3 \\ | \\ \alpha_1 \text{---} \beta_i^{(s)} \text{---} \alpha_4 \end{array} \end{aligned}$$

- ▶ diagonal monodromy around $z = 0 \rightarrow K_{12}^{(s)} = K_{21}^{(s)} = 0$
- ▶ the element of K are related to **3-point functions**

$$K_{11}^{(s)} = C(\alpha_4, \alpha_3, \beta_1^{(s)}) C(Q_0 - \beta_1, -b_0/2, \alpha_1)$$

$$K_{22}^{(s)} = C(\alpha_4, \alpha_3, \beta_2^{(s)}) C(Q_0 - \beta_2, -b_0/2, \alpha_1)$$

where the **internal states** are $\beta_1^{(s)} = \alpha_1 - \frac{b_0}{2}$, $\beta_2^{(s)} = \alpha_1 + \frac{b_0}{2}$

Around $z = 0$

$$I_1^{(s)}(z) = {}_2F_1(a, b; c; z), \quad I_2^{(s)}(z) = z^{1-c} {}_2F_1(1+a-c, 1+b-c; 2-c; z)$$

this is the **s-channel**

$$\begin{aligned} \langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{\alpha_2}(z) V_{\alpha_1}(0) \rangle &\sim \sum_{i,j=1}^2 I_i^{(s)}(\bar{z}) K_{ij}^{(s)} I_j^{(s)}(z) = \sum_i K_{ii}^{(s)} \left\| I_i^{(s)}(z) \right\|^2 \\ &= \sum_i \begin{array}{c} \alpha_3 \\ | \\ -b_0/2 \\ | \\ \alpha_1 \quad \beta_i^{(s)} \quad \alpha_4 \end{array} \end{aligned}$$

- ▶ diagonal monodromy around $z = 0 \rightarrow K_{12}^{(s)} = K_{21}^{(s)} = 0$
- ▶ the element of K are related to **3-point functions**

$$K_{11}^{(s)} = C(\alpha_4, \alpha_3, \beta_1^{(s)}) C(Q_0 - \beta_1, -b_0/2, \alpha_1)$$

$$K_{22}^{(s)} = C(\alpha_4, \alpha_3, \beta_2^{(s)}) C(Q_0 - \beta_2, -b_0/2, \alpha_1)$$

where the **internal states** are $\beta_1^{(s)} = \alpha_1 - \frac{b_0}{2}$, $\beta_2^{(s)} = \alpha_1 + \frac{b_0}{2}$

- ▶ pairing

$$\left\| f(a, b, c, z) \right\|^2 = f(a, b, c, z) f(a, b, c, \bar{z})$$

Around $z = \infty$

$$I_1^{(u)}(z) = z^{-a} {}_2F_1(a, 1 + a - c; 1 + a - b; z^{-1})$$

$$I_2^{(u)}(z) = z^{-b} {}_2F_1(b, 1 + b - c; 1 + b - a; z^{-1})$$

this is the u -channel

Around $z = \infty$

$$I_1^{(u)}(z) = z^{-a} {}_2F_1(a, 1 + a - c; 1 + a - b; z^{-1})$$

$$I_2^{(u)}(z) = z^{-b} {}_2F_1(b, 1 + b - c; 1 + b - a; z^{-1})$$

this is the u -channel

$$\begin{aligned} \langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{\alpha_2}(z) V_{\alpha_1}(0) \rangle &\sim \sum_{i,j=1}^2 I_i^{(u)}(\bar{z}) K_{ij}^{(u)} I_j^{(u)}(z) = \sum_i K_{ii}^{(u)} \left\| I_i^{(u)}(z) \right\|^2 \\ &= \sum_i \begin{array}{c} \alpha_4 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \alpha_1 \text{---} \text{---} \alpha_3 \\ \beta_i^{(u)} \end{array} \end{aligned}$$

Around $z = \infty$

$$I_1^{(u)}(z) = z^{-a} {}_2F_1(a, 1 + a - c; 1 + a - b; z^{-1})$$

$$I_2^{(u)}(z) = z^{-b} {}_2F_1(b, 1 + b - c; 1 + b - a; z^{-1})$$

this is the u -channel

$$\begin{aligned} \langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{\alpha_2}(z) V_{\alpha_1}(0) \rangle &\sim \sum_{i,j=1}^2 I_i^{(u)}(\bar{z}) K_{ij}^{(u)} I_j^{(u)}(z) = \sum_i K_{ii}^{(u)} \left\| I_i^{(u)}(z) \right\|^2 \\ &= \sum_i \text{diagram} \end{aligned}$$

Conformal Bootstrap:

$$\sum_i \text{diagram} = \text{diagram}$$

extending a set of solutions by analytic continuation $I_i^{(s)} = \sum_{j=1}^2 M_{ij} I_j^{(u)}$, we obtain

$$\sum_{k,l=1}^2 K_{kl}^{(s)} M_{ki} M_{lj} = K_{ij}^{(u)}$$

This set of equations determines the 3-point functions.

- ▶ off-diagonal elements

$$\frac{K_{22}^{(s)}}{K_{11}^{(s)}} = -\frac{M_{11}M_{12}}{M_{21}M_{22}}$$

- ▶ diagonal elements

$$\frac{K_{22}^{(u)}}{K_{22}^{(s)}} = (M_{22})^2 - \frac{M_{21}M_{12}M_{22}}{M_{11}} = \frac{M_{22}}{M_{11}}(\det M)$$

- ▶ off-diagonal elements

$$\frac{K_{22}^{(s)}}{K_{11}^{(s)}} = -\frac{M_{11}M_{12}}{M_{21}M_{22}}$$

- ▶ diagonal elements

$$\frac{K_{22}^{(u)}}{K_{22}^{(s)}} = (M_{22})^2 - \frac{M_{21}M_{12}M_{22}}{M_{11}} = \frac{M_{22}}{M_{11}}(\det M)$$

The DOZZ formula solves both the equations

$$C(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{\Upsilon(2\alpha_T - Q_0)} \prod_{r=1}^3 \frac{\Upsilon(2\alpha_r)}{\Upsilon(2\alpha_T - 2\alpha_r)}$$

where $2\alpha_T = \alpha_1 + \alpha_2 + \alpha_3$ and

$$\Upsilon(X) \propto \prod_{n_1, n_2=0}^{\infty} (X + n_1 b_0 + n_2 1/b_0) (-X + (n_1 + 1)b_0 + (n_2 + 1)1/b_0)$$

Conformal Bootstrap approach: 3-point function is derived exploiting symmetries, without using the Lagrangian.

Comments on the relation: $Z_{S^2}^{SQED} = \langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{-b/2}(z) V_{\alpha_1}(0) \rangle$

- ▶ vortices moduli space are slices of instantons moduli space. Indeed
vortex partition functions \Leftrightarrow degenerate conformal blocks
- ▶ flop transformations correspond to different channel decomposition of the correlator.
flop symmetry invariance of $Z_{S^2} \Leftrightarrow$ crossing symmetry

Comments on the relation: $Z_{S^2}^{SQED} = \langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{-b/2}(z) V_{\alpha_1}(0) \rangle$

- ▶ vortices moduli space are slices of instantons moduli space. Indeed
vortex partition functions \Leftrightarrow degenerate conformal blocks
- ▶ flop transformations correspond to different channel decomposition of the correlator.
flop symmetry invariance of Z_{S^2} \Leftrightarrow crossing symmetry

S^2 partition functions and degenerate Liouville correlators satisfy the same constraints thus they are equivalent.

Comments on the relation: $Z_{S^2}^{SQED} = \langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{-b/2}(z) V_{\alpha_1}(0) \rangle$

- ▶ vortices moduli space are slices of instantons moduli space. Indeed
vortex partition functions \Leftrightarrow degenerate conformal blocks
- ▶ flop transformations correspond to different channel decomposition of the correlator.
flop symmetry invariance of $Z_{S^2} \Leftrightarrow$ crossing symmetry

S^2 partition functions and degenerate Liouville correlators satisfy the same constraints thus they are equivalent.

We will now argue that a similar story holds in 3d. We start by reviewing 3d partition functions.

$\mathcal{N} = 2$ theory on S_b^3

$$S_b^3 : b^2 |z_1|^2 + \frac{1}{b^2} |z_2|^2 = 1$$

Coulomb branch localization scheme [Hama-Hosomichi-Lee].

SQED: $U(1)$ gauge group, N_f chirals m_j , N_f anti-chirals \tilde{m}_k , with FI ξ .

$$Z_S^{SQED} = \int dx G_{cl} \cdot G_{1-loop} = \int dx e^{2\pi i x \xi} \prod_{j,k}^{N_f} \frac{s_b(x + m_j + iQ/2)}{s_b(x + \tilde{m}_k - iQ/2)}$$

The 1-loop contribution of a chiral multiplet is:

$$s_b(x) = \prod_{m,n \in \mathbb{Z}_{\geq 0}} \frac{mb + nb^{-1} + \frac{Q}{2} - ix}{mb + nb^{-1} + \frac{Q}{2} + ix}, \quad Q = b + 1/b.$$

$\mathcal{N} = 2$ theory on S_b^3

$$S_b^3 : b^2 |z_1|^2 + \frac{1}{b^2} |z_2|^2 = 1$$

Coulomb branch localization scheme [Hama-Hosomichi-Lee].

SQED: $U(1)$ gauge group, N_f chirals m_j , N_f anti-chirals \tilde{m}_k , with FI ξ .

$$Z_S^{SQED} = \int dx G_{cl} \cdot G_{1-loop} = \int dx e^{2\pi i x \xi} \prod_{j,k} \frac{s_b(x + m_j + iQ/2)}{s_b(x + \tilde{m}_k - iQ/2)}$$

The 1-loop contribution of a chiral multiplet is:

$$s_b(x) = \prod_{m,n \in \mathbb{Z}_{\geq 0}} \frac{mb + nb^{-1} + \frac{Q}{2} - ix}{mb + nb^{-1} + \frac{Q}{2} + ix}, \quad Q = b + 1/b.$$

Flop Symmetry: Z_S^{SQED} is invariant under : $m_j \leftrightarrow -\tilde{m}_k$ and $\xi \leftrightarrow -\xi$
this signs flip exchanges **phase I** and **phase II**

Higgs-branch-like factorized form: [Pasquetti]

$$Z_S^{SQED} = \sum_i^{N_f} G_{cl}^{(i)} G_{1-loop}^{(i)} \left\| \mathcal{Z}_V^{(i)} \right\|_S^2,$$

- ▶ $G_{cl}^{(i)} G_{1-loop}^{(i)}$ are evaluated on Higgs vacua $x = -m_i$

$$G_{cl}^{(i)} = e^{-2\pi i \xi m_i}, \quad G_{1-loop}^{(i)} = \prod_{j,k}^{N_f} \frac{s_b(m_j - m_i + iQ/2)}{s_b(\tilde{m}_k - m_i - iQ/2)},$$

- ▶ 3d vortices are the **K-theory lift** (q-deformation) of 2d vortices:

$$\mathcal{Z}_V^{(i)} = \sum_n \prod_{j,k}^{N_f} \frac{(y_k x_i^{-1}; q)_n}{(q x_j x_i^{-1}; q)_n} z^n = N_f \Phi_{N_f-1}^{(i)}(\vec{x}, \vec{y}; z).$$

- ▶ Vortices are glued with **S-pairing**:

$$\left\| f(x; q) \right\|_S^2 = f(x; q) f(\tilde{x}; \tilde{q})$$

$$x_i = e^{2\pi b m_i}, \quad y_i = e^{2\pi b \tilde{m}_i}, \quad z = e^{2\pi b \xi}, \quad q = e^{2\pi i b^2}, \\ \tilde{x}_i = e^{2\pi m_i/b}, \quad \tilde{y}_i = e^{2\pi \tilde{m}_i/b}, \quad \tilde{z} = e^{2\pi \xi/b}, \quad \tilde{q} = e^{2\pi i/b^2}$$

$\mathcal{N} = 2$ theory on $S^2 \times S^1$

Computes the (generalized) super-conformal-index

[Imamura-Yokoyama],[Kapustin-Willet],[Dimofte-Gukov-Gaiotto].

SQED with fugacities:

$$\begin{aligned}(\phi_i, r_i), & \quad i = 1, \dots, N_f, & \text{flavor } U(1)^{N_f}, \\(\xi_i, l_i), & \quad i = 1, \dots, N_f, & \text{(anti) - flavor } U(1)^{N_f}, \\(\omega, n), & & \text{topological } U(1), \\(t, s), & & \text{gauged } U(1).\end{aligned}$$

$$Z_{id} = \sum_{s \in \mathbb{Z}} \int \frac{dt}{2\pi it} t^n \omega^s \prod_{j=1}^{N_f} \chi(t\phi_j, s + r_j) \prod_{k=1}^{N_f} \chi(t^{-1}\xi_k^{-1}, -s - l_k).$$

The 1-loop contribution of a chiral multiplet is:

$$\chi(\zeta, m) = (q^{1/2}\zeta^{-1})^{-m/2} \prod_{k=0}^{\infty} \frac{(1 - q^{l+1}\zeta^{-1}q^{-m/2})}{(1 - q^l\zeta q^{-m/2})}$$

$\mathcal{N} = 2$ theory on $S^2 \times S^1$

Computes the (generalized) super-conformal-index

[Imamura-Yokoyama],[Kapustin-Willet],[Dimofte-Gukov-Gaiotto].

SQED with fugacities:

$$\begin{aligned}(\phi_i, r_j), & \quad i = 1, \dots, N_f, & \text{flavor } U(1)^{N_f}, \\(\xi_i, l_j), & \quad i = 1, \dots, N_f, & \text{(anti) - flavor } U(1)^{N_f}, \\(\omega, n), & & \text{topological } U(1), \\(t, s), & & \text{gauged } U(1).\end{aligned}$$

$$Z_{id} = \sum_{s \in \mathbb{Z}} \int \frac{dt}{2\pi i t} t^n \omega^s \prod_{j=1}^{N_f} \chi(t \phi_j, s + r_j) \prod_{k=1}^{N_f} \chi(t^{-1} \xi_k^{-1}, -s - l_k).$$

The 1-loop contribution of a chiral multiplet is:

$$\chi(\zeta, m) = (q^{1/2} \zeta^{-1})^{-m/2} \prod_{k=0}^{\infty} \frac{(1 - q^{l+1} \zeta^{-1} q^{-m/2})}{(1 - q^l \zeta q^{-m/2})}$$

Flop Symmetry:

Z_{id}^{SQED} is invariant under : $\omega \leftrightarrow \omega^{-1}$, $n \leftrightarrow -n$, $\phi_j \leftrightarrow \xi_j^{-1}$, $r_j \leftrightarrow -l_j$
exchanges **phase I** and **phase II**

Higgs-branch-like factorized form: [Beem-Dimofte-Pasquetti]

$$Z_{id}^{SQED} = \sum_{i=1}^{N_f} G_{cl}^{(i)} G_{1-loop}^{(i)} \left\| Z_V^i \right\|_{id}^2,$$

- ▶ $G_{cl}^{(i)} G_{1-loop}^{(i)}$ are evaluated on Higgs vacua $t = \phi_i^{-1}$, $s = -r_i$:

$$G_{cl}^{(i)} = \omega^{-r_i} (\phi_i^{-1})^n, \quad G_{1-loop}^{(i)} = \prod_{j=1}^{N_f} \chi(\phi_j \phi_i^{-1}, r_j - r_i) \prod_{k=1}^{N_f} \chi(\phi_i \xi_k^{-1}, r_i - l_k),$$

- ▶ 3d vortices are the K-theory lift (q-deformation) of 2d vortices:

$$Z_V^{(i)} = \sum_n \prod_{j,k}^{N_f} \frac{(y_k x_i^{-1}; q)_n}{(q x_j x_i^{-1}; q)_n} z^n = N_f \Phi_{N_f-1}^{(i)}(\vec{x}, \vec{y}; z).$$

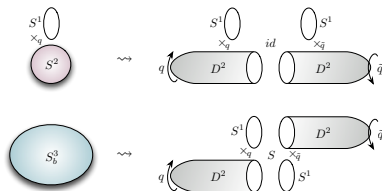
- ▶ Vortices are glued with *id-pairing*:

$$\left\| f(x; q) \right\|_{id}^2 := f(x; q) f(\tilde{x}; \tilde{q}),$$

$$\begin{aligned} x_i &= \phi_i q^{r_i/2}, & \tilde{x}_i &= \phi_i^{-1} q^{r_i/2}, & y_i &= \xi_i q^{l_i/2}, & \tilde{y}_i &= \xi_i^{-1} q^{l_i/2}, \\ z &= \omega q^{n/2}, & \tilde{q} &= q^{-1} \end{aligned}$$

Comment on *id*-pairing and *S*-pairing

[Beem-Dimofte-Pasquetti]



S_b^3 and $S^1 \times S^2$ geometries obtained gluing solid tori

- ▶ **id-fusion:** $S^1 \times S^2$
- ▶ **S-fusion:** S_b^3

Flop symmetry

- ▶ On the **Coulomb branch** it's a trivial symmetry of the integrand
- ▶ On the **Higgs branch** it translates into highly non-trivial relations:

$$\begin{aligned} Z_{id}^I &= \sum_i^{N_f} G_{cl}^{(i),I} G_{1loop}^{(i),I} \left\| \mathcal{Z}_V^{(i),I} \right\|_{id}^2 = \\ &= \sum_i^{N_f} G_{cl}^{(i),II} G_{1loop}^{(i),II} \left\| \mathcal{Z}_V^{(i),II} \right\|_{id}^2 = Z_{id}^{II}, \end{aligned}$$

$$\begin{aligned} Z_S^I &= \sum_i^{N_f} G_{cl}^{(i),I} G_{1loop}^{(i),I} \left\| \mathcal{Z}_V^{(i),I} \right\|_S^2 = \\ &= \sum_i^{N_f} G_{cl}^{(i),II} G_{1loop}^{(i),II} \left\| \mathcal{Z}_V^{(i),II} \right\|_S^2 = Z_S^{II}. \end{aligned}$$

⇒ Can these relations constrain the 1-loop part of the partition function?

3d partition functions and q -deformed CFT

3d partition functions, in the Higgs branch expression, have a structure similar to degenerate Liouville correlators

$$Z_{id,S} = \sum_{i=1}^{N_f} G_{cl}^{(i)} G_{1loop}^{(i)} \left\| Z_V^i \right\|_{id,S}^2$$

- ▶ gauge theory flop symmetry \Rightarrow crossing symmetry
- ▶ q -deformed hypers as conformal blocks \Rightarrow q -deformation of Virasoro

We will construct q -deformed degenerate correlators using the bootstrap approach. These correlators are equivalent to 3d partition functions.

q -deformed Virasoro algebra $\mathcal{V}ir_{q,t}$

$\mathcal{V}ir_{q,t}$ has two complex parameters q, t (useful to consider $p = \frac{q}{t}$) and an infinite set of generators T_n with $n \in \mathbb{Z}$ [Shiraishi, Kubo, Awata, Odake]

$$[T_n, T_m] = - \sum_{l=1}^{+\infty} f_l (T_{n-l} T_{m+l} - T_{m-l} T_{n+l}) - \frac{(1-q)(1-t^{-1})}{1-p} (p^n - p^{-n}) \delta_{m+n,0}$$

where

$$f(z) = \sum_{l=0}^{+\infty} f_l z^l = \exp \left[\sum_{l=1}^{+\infty} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})}{1+p^n} z^n \right]$$

- ▶ invariant under: $(q, t) \rightarrow (q^{-1}, t^{-1})$ and $(q, t) \rightarrow (t, q)$
- ▶ considering: $t = q^{-b_0^2}$ and $q \rightarrow 1$

$\mathcal{V}ir_{q,t}$ reduces to the Virasoro with central charge $c_V = 1 + 6Q_0^2$

$(T(z) = \sum_n T_n z^{-n})$ reduces to the Virasoro current $L(z) = \sum_n L_n z^{-n-2}$

Representations of $\mathcal{V}ir_{q,t}$ can be constructed using Verma modules

- ▶ The highest weight state $|\lambda\rangle$ satisfies

$$T_0|\lambda\rangle = \lambda|\lambda\rangle, \quad T_n|\lambda\rangle = 0 \quad \text{for } n > 0$$

- ▶ the Verma module $\mathcal{M}(\lambda)$ is constructed acting on the highest weight state $|\lambda\rangle$ with the operators T_{-n} with $n > 0$
- ▶ Verma modules can include singular states. In particular, there is a level 2 singular vector for the following values of the parameter λ

$$\lambda_1 = p^{1/2}q^{1/2} + p^{-1/2}q^{-1/2}, \quad \lambda_2 = p^{1/2}t^{-1/2} + p^{-1/2}t^{1/2}$$

Conformal blocks with 3 non-degenerate and 1 degenerate insertion satisfy **difference equations** [Awata, Kubo, Morita, Odake, Shiraishi] [Awata, Yamada][Schiappa, Wyllard]

q-deformed CFT

q-deformed Bootstrap Approach: Consider degenerate conformal blocks, constrained to satisfy difference equations. Impose crossing symmetry to derive 3-point functions.

- ▶ 4-point function with a **degenerate insertion**

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(r) V_{\alpha_2}(z, \tilde{z}) V_{\alpha_1}(0) \rangle \sim G(z, \tilde{z})$$

- ▶ $V_{\alpha_2}(z, \tilde{z})$ has a null state at level 2, leading to

$$D(A, B; C; q; z)G(z, z) = 0, \quad D(\tilde{A}, \tilde{B}; \tilde{C}; \tilde{q}; \tilde{z})G(z, \tilde{z}) = 0,$$

where $D(A, B; C; q; z)$ is the q -hypergeometric operator

$$D(A, B; C; q; z) = h_2 \frac{\partial_q^2}{\partial_q z^2} + h_1 \frac{\partial_q}{\partial_q z} + h_0$$

and $\frac{\partial_q}{\partial_q z}$ is the q -derivative. It acts on a function $f(z)$ as

$$\frac{\partial_q}{\partial_q z} f(z) = \frac{f(qz) - f(z)}{z(q-1)}$$

$G(z, \tilde{z})$ is a bilinear combination of **solutions of the q -hypergeometric eq.**

Around $z = 0$

$$I_1^{(s)} = F_1^{(s)}(z), \quad I_2^{(s)} = T_2^{(s)} F_2^{(s)}(z)$$

where:

▶ $F_1^{(s)}(z) = {}_2\Phi_1(A, B; C; z)$

$$F_2^{(s)}(z) = {}_2\Phi_1(qAC^{-1}, qBC^{-1}; q^2C^{-1}; z)$$

▶ $T_2^{(s)} := T_{qC^{-1}}(z^{-1}r^{1/2}q)$

where we defined the twist function $T_A(u) = \frac{\theta(Au; q)}{\theta(A; q)\theta(u; q)}$

Around $z = 0$

$$I_1^{(s)} = F_1^{(s)}(z), \quad I_2^{(s)} = T_2^{(s)} F_2^{(s)}(z)$$

where:

▶ $F_1^{(s)}(z) = {}_2\Phi_1(A, B; C; z)$

$$F_2^{(s)}(z) = {}_2\Phi_1(qAC^{-1}, qBC^{-1}; q^2C^{-1}; z)$$

▶ $T_2^{(s)} := T_{qC^{-1}}(z^{-1}r^{1/2}q)$

where we defined the twist function $T_A(u) = \frac{\theta(Au; q)}{\theta(A; q)\theta(u; q)}$

- ▶ **Virasoro limit:** $\lim_{q \rightarrow 1} {}_2\Phi_1(q^a, q^b; q^c; q, z) = {}_2F_1(a, b; c; z)$
and $\lim_{q \rightarrow 1} T_A(u) = u^{-a}$, s-channel basis discussed above

Around $z = 0$

$$I_1^{(s)} = F_1^{(s)}(z), \quad I_2^{(s)} = T_2^{(s)} F_2^{(s)}(z)$$

where:

$$\blacktriangleright F_1^{(s)}(z) = {}_2\Phi_1(A, B; C; z)$$

$$F_2^{(s)}(z) = {}_2\Phi_1(qAC^{-1}, qBC^{-1}; q^2C^{-1}; z)$$

$$\blacktriangleright T_2^{(s)} := T_{qC^{-1}}(z^{-1}r^{1/2}q)$$

where we defined the twist function $T_A(u) = \frac{\theta(Au; q)}{\theta(A; q)\theta(u; q)}$

- Virasoro limit:** $\lim_{q \rightarrow 1} {}_2\Phi_1(q^a, q^b; q^c; q, z) = {}_2F_1(a, b; c; z)$
and $\lim_{q \rightarrow 1} T_A(u) = u^{-a}$, s-channel basis discussed above

so in the s-channel

$$\begin{aligned} \langle V_{\alpha_4}(\infty) V_{\alpha_3}(r) V_{\alpha_2}(z) V_{\alpha_1}(0) \rangle &\sim \sum_{i,j=1}^2 \tilde{l}_i^{(s)}(\tilde{z}; \tilde{q}) K_{ij}^{(s)} I_j^{(s)}(z; q) \\ &= \sum_{i=1}^2 K_{ii}^{(s)} \left\| I_i^{(s)}(z; q) \right\|_*^2 = \sum_i \begin{array}{c} \alpha_2 \quad \alpha_3 \\ \vdots \quad | \\ \hline \alpha_1 \quad \alpha_4 \end{array} \end{aligned}$$

- generic pairing

$$\left\| f(A, B, C; z; q) \right\|_*^2 = f(A, B, C; z; q) f(\tilde{A}, \tilde{B}, \tilde{C}; \tilde{z}; \tilde{q})$$

Around $z = \infty$

$$I_1^{(u)} = T_1^{(u)} F_1^{(u)}(z^{-1} r q^2), \quad I_2^{(u)} = T_2^{(u)} F_2^{(u)}(z^{-1} r q^2)$$

where

- ▶ $F_1^{(u)}(z^{-1} r q^2) = {}_2\Phi_1(A, qAC^{-1}; qAB^{-1}; q^2 r z^{-1})$
 $F_2^{(u)}(z^{-1} r q^2) = {}_2\Phi_1(B, qBC^{-1}; qBA^{-1}; q^2 r z^{-1})$
- ▶ the twist functions are given by
 $T_1^{(u)} := T_{A^{-1}}(z^{-1} r^{1/2} q), \quad T_2^{(u)} := T_{B^{-1}}(z^{-1} r^{1/2} q)$
- ▶ in the $q \rightarrow 1$ limit we recover the undeformed u -channel basis

Around $z = \infty$

$$I_1^{(u)} = T_1^{(u)} F_1^{(u)}(z^{-1} r q^2), \quad I_2^{(u)} = T_2^{(u)} F_2^{(u)}(z^{-1} r q^2)$$

where

$$\blacktriangleright F_1^{(u)}(z^{-1} r q^2) = {}_2\Phi_1(A, qAC^{-1}; qAB^{-1}; q^2 r z^{-1})$$

$$F_2^{(u)}(z^{-1} r q^2) = {}_2\Phi_1(B, qBC^{-1}; qBA^{-1}; q^2 r z^{-1})$$

\blacktriangleright the twist functions are given by

$$T_1^{(u)} := T_{A^{-1}}(z^{-1} r^{1/2} q), \quad T_2^{(u)} := T_{B^{-1}}(z^{-1} r^{1/2} q)$$

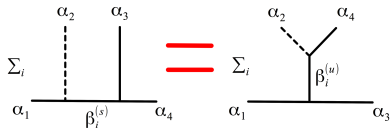
\blacktriangleright in the $q \rightarrow 1$ limit we recover the undeformed u -channel basis

The correlation function in the u -channel is

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(r) V_{\alpha_2}(z) V_{\alpha_1}(0) \rangle \sim \sum_{i,j=1}^2 \tilde{I}_i^{(u)}(\tilde{z}; \tilde{q}) K_{ij}^{(u)} I_j^{(s)}(z; q)$$

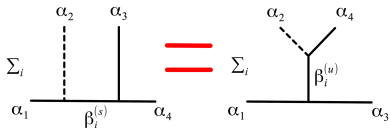
$$= \sum_{i=1}^2 K_{ii}^{(u)} \left\| I_i^{(u)}(z; q) \right\|_*^2 = \sum_i \begin{array}{c} \alpha_2 \quad \alpha_4 \\ \diagdown \quad / \\ \beta_i^{(u)} \\ | \\ \alpha_1 \quad \alpha_3 \end{array}$$

q-deformed Bootstrap:



$$K_{11}^{(s)} \left\| \left\| I_1^{(s)} \right\|_* \right\|^2 + K_{22}^{(s)} \left\| \left\| I_2^{(s)} \right\|_* \right\|^2 = K_{11}^{(u)} \left\| \left\| I_1^{(u)} \right\|_* \right\|^2 + K_{22}^{(u)} \left\| \left\| I_2^{(u)} \right\|_* \right\|^2$$

q-deformed Bootstrap:



$$K_{11}^{(s)} \left\| \left\| I_1^{(s)} \right\|_* \right\|^2 + K_{22}^{(s)} \left\| \left\| I_2^{(s)} \right\|_* \right\|^2 = K_{11}^{(u)} \left\| \left\| I_1^{(u)} \right\|_* \right\|^2 + K_{22}^{(u)} \left\| \left\| I_2^{(u)} \right\|_* \right\|^2$$

using analytical continuation $I_i^{(s)} = \sum_{j=1}^2 M_{ij} I_j^{(u)}$ and $\tilde{I}_i^{(s)} = \sum_{j=1}^2 \tilde{M}_{ij} \tilde{I}_j^{(u)}$ we obtain

$$\sum_{k,l=1}^2 K_{kl}^{(s)} \tilde{M}_{ki} M_{lj} = K_{ij}^{(u)}$$

Bootstrap equation that determines the 3-point functions. In details

▶ off-diagonal elements $\frac{K_{22}^{(s)}}{K_{11}^{(s)}} = -\frac{\tilde{M}_{11} M_{12}}{\tilde{M}_{21} M_{22}}$

▶ diagonal elements $\frac{K_{22}^{(u)}}{K_{22}^{(s)}} = \frac{M_{22}}{\tilde{M}_{11}} (\det \tilde{M})$

solutions to these equations exist only for certain pairings!

id-pairing q -CFT

Consider the case where the conformal blocks are glued as

$$\left\| f(A, B, C; z; q) \right\|_{\text{id}}^2 = f(A, B, C; z; q) f(\tilde{A}, \tilde{B}, \tilde{C}; \tilde{z}; \tilde{q}).$$

▶ with:

$$\begin{aligned} A &= e^{\beta X_A}, & B &= e^{\beta X_B}, & C &= q e^{\beta X_C}, & q &= e^{\beta/b_0}, & z \\ \tilde{A} &= e^{-\beta X_A}, & \tilde{B} &= e^{-\beta X_B}, & \tilde{C} &= \tilde{q} e^{-\beta X_C}, & \tilde{q} &= q^{-1}, & \tilde{z} &= \bar{z} \end{aligned}$$

▶ then choose:

$$\begin{aligned} X_A &= \alpha_1 + \alpha_3 + \alpha_4 - \frac{b_0}{2} - Q_0, & X_B &= \alpha_1 + \alpha_3 - \alpha_4 - \frac{b_0}{2}, \\ X_C &= 2\alpha_1 - Q_0, \end{aligned}$$

id-pairing q -CFT

Consider the case where the conformal blocks are glued as

$$\left\| f(A, B, C; z; q) \right\|_{\text{id}}^2 = f(A, B, C; z; q) f(\tilde{A}, \tilde{B}, \tilde{C}; \tilde{z}; \tilde{q}).$$

► with:

$$\begin{aligned} A &= e^{\beta X_A}, & B &= e^{\beta X_B}, & C &= q e^{\beta X_C}, & q &= e^{\beta/b_0}, & z & \\ \tilde{A} &= e^{-\beta X_A}, & \tilde{B} &= e^{-\beta X_B}, & \tilde{C} &= \tilde{q} e^{-\beta X_C}, & \tilde{q} &= q^{-1}, & \tilde{z} &= \bar{z} \end{aligned}$$

► then choose:

$$\begin{aligned} X_A &= \alpha_1 + \alpha_3 + \alpha_4 - \frac{b_0}{2} - Q_0, & X_B &= \alpha_1 + \alpha_3 - \alpha_4 - \frac{b_0}{2}, \\ X_C &= 2\alpha_1 - Q_0, \end{aligned}$$

The bootstrap equations are solved by

$$C_{id}(\alpha_3, \alpha_2, \alpha_1) = \frac{1}{\Upsilon^\beta(2\alpha_T - Q_0)} \prod_{i=1}^3 \frac{\Upsilon^\beta(2\alpha_i)}{\Upsilon^\beta(2\alpha_T - 2\alpha_i)}$$

► where we defined: $\Upsilon^\beta(X) \propto \prod_{k=-\infty}^{+\infty} \Upsilon\left(X + i\frac{2\pi}{\beta}k\right)$

$S^1 \times S^2$ superconformal index for SQED is equivalent to an id -pairing q -CFT degenerate correlator ($\alpha_2 = -b_0/2$)

$$Z_{id} = \sum_{i=1}^{N_f} G_{cl}^{(i),I} G_{1loop}^{(i),I} \left\| Z_V^{(i),I} \right\|_{id}^2 \sim \sum_{i=1}^2 K_{ii}^{(s)} \left\| I_i^{(s)}(z; q) \right\|_{id}^2 = \sum_i \begin{array}{c} \alpha_2 \quad \alpha_3 \\ | \quad | \\ \hline \alpha_1 \quad \alpha_4 \end{array}$$

$S^1 \times S^2$ superconformal index for SQED is equivalent to an id -pairing q -CFT degenerate correlator ($\alpha_2 = -b_0/2$)

$$Z_{id} = \sum_{i=1}^{N_f} G_{cl}^{(i),I} G_{1loop}^{(i),I} \left\| Z_V^{(i),I} \right\|_{id}^2 \sim \sum_{i=1}^2 K_{ii}^{(s)} \left\| I_i^{(s)}(z; q) \right\|_{id}^2 = \sum_i \begin{array}{c} \alpha_2 \quad \alpha_3 \\ | \quad | \\ \hline \alpha_1 \quad \alpha_4 \end{array}$$

dictionary: writing the flavor fugacities as $\phi_i = e^{i\beta \Phi_i}$, $\xi_i = e^{i\beta \Xi_i}$

$$\alpha_1 = \frac{Q_0}{2} + i \frac{\Phi_1 - \Phi_2}{2}, \quad \alpha_3 = \frac{b_0}{2} - i \frac{\Xi_1 + \Xi_2 - \Phi_1 - \Phi_2}{2}, \quad \alpha_4 = \frac{Q_0}{2} - i \frac{\Xi_1 - \Xi_2}{2},$$

where $\beta = \text{circumference of } S^1$

$S^1 \times S^2$ superconformal index for SQED is equivalent to an id -pairing q -CFT degenerate correlator ($\alpha_2 = -b_0/2$)

$$Z_{id} = \sum_{i=1}^{N_f} G_{cl}^{(i),I} G_{1loop}^{(i),I} \left\| Z_V^{(i),I} \right\|_{id}^2 \sim \sum_{i=1}^2 K_{ii}^{(s)} \left\| I_i^{(s)}(z; q) \right\|_{id}^2 = \sum_i \begin{array}{c} \alpha_2 \\ \alpha_1 \text{---} \text{---} \alpha_4 \\ \alpha_3 \end{array}$$

dictionary: writing the flavor fugacities as $\phi_i = e^{i\beta \Phi_i}$, $\xi_i = e^{i\beta \Xi_i}$

$$\alpha_1 = \frac{Q_0}{2} + i \frac{\Phi_1 - \Phi_2}{2}, \quad \alpha_3 = \frac{b_0}{2} - i \frac{\Xi_1 + \Xi_2 - \Phi_1 - \Phi_2}{2}, \quad \alpha_4 = \frac{Q_0}{2} - i \frac{\Xi_1 - \Xi_2}{2},$$

where $\beta = \text{circumference of } S^1$

- ▶ gauge theory flop symmetry \Leftrightarrow q -CFT crossing symmetry
- ▶ $\beta \rightarrow 0$ limit
 - ▶ CFT: $\mathcal{Vir}_{q,t} \rightarrow$ Virasoro, we recover Liouville theory results
 - ▶ gauge: $S^1 \times S^2$ partition function reduce to S^2 partition function

Consistent: S^2 partition functions match degenerate Liouville correlators

S-pairing q -CFT

Now, consider the case where the conformal blocks are glued as

$$\left\| f(A, B, C; z; q) \right\|_S^2 = f(A, B, C; z; q) f(\tilde{A}, \tilde{B}, \tilde{C}; \tilde{z}; \tilde{q}).$$

► where

$$A = e^{2\pi i X_A / \omega_2}, \quad B = e^{2\pi i X_B / \omega_2}, \quad C = e^{2\pi i X_C / \omega_2}, \quad q = e^{2\pi i \frac{\omega_1}{\omega_2}}, \quad z = e^{2\pi i Z / \omega_2}$$
$$\tilde{A} = e^{2\pi i X_A / \omega_1}, \quad \tilde{B} = e^{2\pi i X_B / \omega_1}, \quad \tilde{C} = e^{2\pi i X_C / \omega_1}, \quad \tilde{q} = e^{2\pi i \frac{\omega_2}{\omega_1}}, \quad \tilde{z} = e^{2\pi i Z / \omega_1}$$

► then choose:

$$X_A = \alpha_1 + \alpha_3 + \alpha_4 - \frac{\omega_3}{2} - E \quad X_B = \alpha_1 + \alpha_3 - \alpha_4 - \frac{\omega_3}{2},$$
$$X_C = 2\alpha_1 - \omega_3 \quad \text{and} \quad E = \omega_1 + \omega_2 + \omega_3$$

S-pairing q -CFT

Now, consider the case where the conformal blocks are glued as

$$\left\| f(A, B, C; z; q) \right\|_S^2 = f(A, B, C; z; q) f(\tilde{A}, \tilde{B}, \tilde{C}; \tilde{z}; \tilde{q}).$$

► where

$$A = e^{2\pi i X_A / \omega_2}, \quad B = e^{2\pi i X_B / \omega_2}, \quad C = e^{2\pi i X_C / \omega_2}, \quad q = e^{2\pi i \frac{\omega_1}{\omega_2}}, \quad z = e^{2\pi i Z / \omega_2}$$
$$\tilde{A} = e^{2\pi i X_A / \omega_1}, \quad \tilde{B} = e^{2\pi i X_B / \omega_1}, \quad \tilde{C} = e^{2\pi i X_C / \omega_1}, \quad \tilde{q} = e^{2\pi i \frac{\omega_2}{\omega_1}}, \quad \tilde{z} = e^{2\pi i Z / \omega_1}$$

► then choose:

$$X_A = \alpha_1 + \alpha_3 + \alpha_4 - \frac{\omega_3}{2} - E \quad X_B = \alpha_1 + \alpha_3 - \alpha_4 - \frac{\omega_3}{2},$$
$$X_C = 2\alpha_1 - \omega_3 \quad \text{and} \quad E = \omega_1 + \omega_2 + \omega_3$$

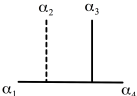
The bootstrap equations are solved by

$$C_S(\alpha_3, \alpha_2, \alpha_1) = \frac{1}{S_3(2\alpha_T - E)} \prod_{i=1}^3 \frac{S_3(2\alpha_i)}{S_3(2\alpha_T - 2\alpha_i)}$$

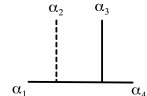
where $S_3(X)$ is the triple sine function

$$S_3(X) \propto \prod_{n_1, n_2, n_3=0}^{+\infty} (\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 + X) (\omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 + E - X)$$

S^3 partition function for SQED is equivalent to an S -pairing q -CFT degenerate correlator ($\alpha_2 = \omega_3/2$)

$$Z_S = \sum_{i=1}^{N_f} G_{cl}^{(i),I} G_{1loop}^{(i),I} \left\| Z_V^{(i),I} \right\|_S^2 \sim \sum_{i=1}^2 K_{ii}^{(s)} \left\| I_i^{(s)}(z; q) \right\|_S^2 = \sum_i$$


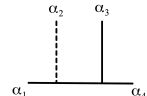
S^3 partition function for SQED is equivalent to an S -pairing q -CFT degenerate correlator ($\alpha_2 = \omega_3/2$)

$$Z_S = \sum_{i=1}^{N_f} G_{cl}^{(i),I} G_{1/loop}^{(i),I} \left\| Z_V^{(i),I} \right\|_S^2 \sim \sum_{i=1}^2 K_{ii}^{(s)} \left\| I_i^{(s)}(z; q) \right\|_S^2 = \sum_i$$


dictionary:

$$\alpha_1 = \frac{E}{2} + j \frac{m_1 - m_2}{2}, \quad \alpha_3 = \frac{\omega_3}{2} - j \frac{\tilde{m}_1 + \tilde{m}_2 - m_1 - m_2}{2}, \quad \alpha_4 = \frac{E}{2} - j \frac{\tilde{m}_1 - \tilde{m}_2}{2},$$

S^3 partition function for SQED is equivalent to an S -pairing q -CFT degenerate correlator ($\alpha_2 = \omega_3/2$)

$$Z_S = \sum_{i=1}^{N_f} G_{cl}^{(i),I} G_{1loop}^{(i),I} \left\| Z_V^{(i),I} \right\|_S^2 \sim \sum_{i=1}^2 K_{ii}^{(s)} \left\| I_i^{(s)}(z; q) \right\|_S^2 = \sum_i$$


dictionary:

$$\alpha_1 = \frac{E}{2} + j \frac{m_1 - m_2}{2}, \quad \alpha_3 = \frac{\omega_3}{2} - j \frac{\tilde{m}_1 + \tilde{m}_2 - m_1 - m_2}{2}, \quad \alpha_4 = \frac{E}{2} - j \frac{\tilde{m}_1 - \tilde{m}_2}{2},$$

- ▶ gauge theory flop symmetry \Leftrightarrow q -CFT crossing symmetry
- ▶ in fact there are 3 possibilities:

$$\alpha_2 = \omega_k/2, \quad q = e^{2\pi i \frac{\omega_i}{\omega_j}}, \quad \tilde{q} = e^{2\pi i \frac{\omega_j}{\omega_i}}, \quad i \neq j \neq k = 1, 2, 3.$$

so far:

3d gauge theory partition functions \Leftrightarrow q -CFT degenerate correlators

$$Z_{S,id} = \sum_{i=1}^{N_f} G_{cl}^{(i),I} G_{1loop}^{(i),I} \left\| Z_V^{(i),I} \right\|_{S,id}^2 \sim \sum_{i=1}^2 K_{ii}^{(s)} \left\| I_i^{(s)}(z; q) \right\|_{S,id}^2 = \sum_i \alpha_1 \begin{array}{c} \alpha_2 \\ \hline \alpha_3 \\ \hline \alpha_4 \end{array}$$

and to show this we derive 3-point functions via bootstrap

so far:

3d gauge theory partition functions \Leftrightarrow q -CFT degenerate correlators

$$Z_{S,id} = \sum_{i=1}^{N_f} G_{cl}^{(i),I} G_{1loop}^{(i),I} \left\| Z_V^{(i),I} \right\|_{S,id}^2 \sim \sum_{i=1}^2 K_{ii}^{(s)} \left\| I_i^{(s)}(z; q) \right\|_{S,id}^2 = \sum_i \begin{array}{c} \alpha_2 \quad \alpha_3 \\ | \quad | \\ \hline \alpha_1 \quad \alpha_4 \end{array}$$

and to show this we derive 3-point functions via bootstrap

Let's now consider non-degenerate correlators

$$\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle_{S,id} = \int d\alpha \begin{array}{c} \alpha_2 \quad \alpha_3 \\ | \quad | \\ \hline \alpha_1 \quad \alpha \quad \alpha_4 \end{array} = \int d\alpha C_{S,id} C_{S,id} (\text{Conf.Blocks})$$

in analogy with the AGT case, it is natural to expect that

5d gauge theory partition functions \Leftrightarrow q -CFT non-degenerate correlators

$\mathcal{N} = 1$ theory on $S^4 \times S^1$

Computes 5d super-conformal index.

Coulomb branch localization yields: [Kim-Kim-Lee],[Terashima]

$$Z_{S^4 \times S^1} = \int d\vec{\sigma} \mathcal{Z}_{1\text{-loop}}(\vec{\sigma}, \vec{m}) \left| Z_{inst}^{5d}(\vec{\sigma}, \vec{m}, \tau; q, t) \right|^2$$

- ▶ $|Z_{inst}^{5d}|^2$ is the contribution of point-like instantons at N and S poles

$$\left| Z_{inst}^{5d}(\vec{\sigma}, \vec{m}, \tau; q, t) \right|^2 = Z_{inst}^{5d}(\vec{\sigma}, \vec{m}, \tau; q, t) Z_{inst}^{5d}(\vec{\sigma}, \vec{m}, \tau; q^{-1}, t^{-1})$$

- ▶ 1-loop contribution
 - ▶ vector multiplet:

$$\mathcal{Z}_{1\text{-loop}}^{\text{vect}}(\sigma) = \prod_{\alpha > 0} \Upsilon^\beta(i\alpha(\sigma)) \Upsilon^\beta(-i\alpha(\sigma))$$

- ▶ hypermultiplet of mass m in a representation R :

$$\mathcal{Z}_{1\text{-loop}}^{\text{hyper}}(\sigma, m, R) = \prod_{\rho \in R} \Upsilon^\beta \left(i(\rho(\sigma) + m) + \frac{Q_0}{2} \right)^{-1}$$

$\beta = \text{circumference of } S^1$

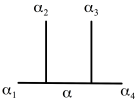
We propose

$Z_{S^4 \times S^1}$ correspond to non-degenerate correlators with $\mathcal{V}ir_{qt} \otimes \mathcal{V}ir_{qt}$ symmetry and *id*-pairing 3-point function.

We propose

$Z_{S^4 \times S^1}$ correspond to non-degenerate correlators with $\mathcal{V}ir_{qt} \otimes \mathcal{V}ir_{qt}$ symmetry and id -pairing 3-point function.

Example:

$$Z_{S^4 \times_q S^1}^{SQCD} = \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle_{id} = \int d\alpha$$


- ▶ 5d instantons vs $\mathcal{V}ir_{qt}$ non-degenerate conformal blocks:

[Awata-Yamada],[Mironov-Morozov-Shakirov-Smirnov]

$$Z_{inst}^{5d, SCQCD} = \mathcal{F}_{\alpha_1 \alpha_2 \alpha \alpha_3 \alpha_4}^{qt}$$

- ▶ 1-loop vs 3-point function:

$$Z_{1\text{-loop}}^{\text{vect}}(\sigma) \prod_{i=1}^4 Z_{1\text{-loop}}^{\text{hyper}}(\sigma, m_i, F) = C_{id}(\alpha_1, \alpha_2, \alpha) C_{id}(Q_0 - \alpha, \alpha_3, \alpha_4)$$

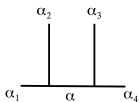
with dictionary:

$$\alpha = i\sigma + \frac{Q_0}{2}, \quad \alpha_1 \pm \alpha_2 = im_{1,2} + Q_0, \quad \alpha_3 \pm \alpha_4 = im_{3,4} + Q_0$$

We propose

$Z_{S^4 \times S^1}$ correspond to non-degenerate correlators with $\mathcal{V}ir_{qt} \otimes \mathcal{V}ir_{qt}$ symmetry and id -pairing 3-point function.

Example:

$$Z_{S^4 \times_q S^1}^{SQCD} = \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle_{id} = \int d\alpha$$


- ▶ 5d instantons vs $\mathcal{V}ir_{qt}$ non-degenerate conformal blocks:

[Awata-Yamada], [Mironov-Morozov-Shakirov-Smirnov]

$$Z_{inst}^{5d, SCQCD} = \mathcal{F}_{\alpha_1 \alpha_2 \alpha \alpha_3 \alpha_4}^{qt}$$

- ▶ 1-loop vs 3-point function:

$$Z_{1\text{-loop}}^{\text{vect}}(\sigma) \prod_{i=1}^4 Z_{1\text{-loop}}^{\text{hyper}}(\sigma, m_i, F) = C_{id}(\alpha_1, \alpha_2, \alpha) C_{id}(Q_0 - \alpha, \alpha_3, \alpha_4)$$

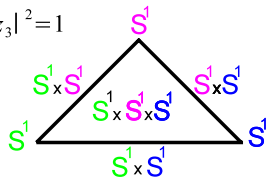
with dictionary:

$$\alpha = i\sigma + \frac{Q_0}{2}, \quad \alpha_1 \pm \alpha_2 = im_{1,2} + Q_0, \quad \alpha_3 \pm \alpha_4 = im_{3,4} + Q_0$$

$S^2 \times S^1$ is a codimension 2 defect \rightarrow degenerate correlator

$\mathcal{N} = 1$ theory on S^5

$$\omega_1 |z_1|^2 + \omega_2 |z_2|^2 + \omega_3 |z_3|^2 = 1$$



$\mathcal{N} = 1$ theory on S^5

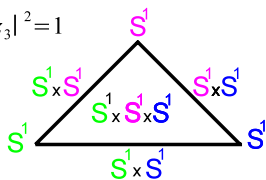
$$\omega_1 |z_1|^2 + \omega_2 |z_2|^2 + \omega_3 |z_3|^2 = 1$$

Coulomb branch localization yields:

[Kallen-Zabzine],[Imamura],[Kim-Kim-Kim],[Lockart-Vafa],

$$Z_{S^5} = \int \prod_i d\sigma_i Z_{cl}(\vec{\sigma}, \tau; \vec{\omega}) Z_{1-loop}(\vec{\sigma}, \vec{m}; \vec{\omega})$$

$$\times Z_{inst}^{5d,I}(\vec{\sigma}, \vec{m}, \tau; q, t) Z_{inst}^{5d,II}(\vec{\sigma}, \vec{m}, \tau; q, t) Z_{inst}^{5d,III}(\vec{\sigma}, \vec{m}, \tau; q, t)$$



$\mathcal{N} = 1$ theory on S^5

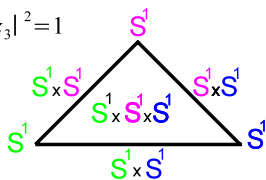
$$\omega_1 |z_1|^2 + \omega_2 |z_2|^2 + \omega_3 |z_3|^2 = 1$$

Coulomb branch localization yields:

[Kallen-Zabzine],[Imamura],[Kim-Kim-Kim],[Lockart-Vafa],

$$Z_{S^5} = \int \prod_i d\sigma_i Z_{cl}(\vec{\sigma}, \tau; \vec{\omega}) Z_{1-loop}(\vec{\sigma}, \vec{m}; \vec{\omega})$$

$$\times Z_{inst}^{5d,I}(\vec{\sigma}, \vec{m}, \tau; q, t) Z_{inst}^{5d,II}(\vec{\sigma}, \vec{m}, \tau; q, t) Z_{inst}^{5d,III}(\vec{\sigma}, \vec{m}, \tau; q, t)$$



▶ 1-loop contribution

▶ vector multiplet:

$$Z_{1-loop}^{\text{vect}}(\sigma) = \prod_{\alpha > 0} S_3(i\alpha(\sigma)) S_3(-i\alpha(\sigma))$$

▶ hypermultiplet of mass m in a representation R :

$$Z_{1-loop}^{\text{hyper}}(\sigma, m, R) = \prod_{\rho \in R} S_3 \left(i(\rho(\sigma) + m) + \frac{E}{2} \right)^{-1},$$

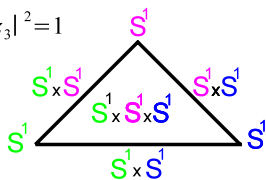
where $E = \omega_1 + \omega_2 + \omega_3$.

$\mathcal{N} = 1$ theory on S^5

$$\omega_1 |z_1|^2 + \omega_2 |z_2|^2 + \omega_3 |z_3|^2 = 1$$

Coulomb branch localization yields:

[Kallen-Zabzine], [Imamura], [Kim-Kim-Kim], [Lockart-Vafa],



$$Z_{S^5} = \int \prod_i d\sigma_i Z_{cl}(\vec{\sigma}, \tau; \vec{\omega}) Z_{1-loop}(\vec{\sigma}, \vec{m}; \vec{\omega}) \\ \times Z_{inst}^{5d,I}(\vec{\sigma}, \vec{m}, \tau; q, t) Z_{inst}^{5d,II}(\vec{\sigma}, \vec{m}, \tau; q, t) Z_{inst}^{5d,III}(\vec{\sigma}, \vec{m}, \tau; q, t)$$

▶ 1-loop contribution

▶ vector multiplet:

$$Z_{1-loop}^{vect}(\sigma) = \prod_{\alpha > 0} S_3(i\alpha(\sigma)) S_3(-i\alpha(\sigma))$$

▶ hypermultiplet of mass m in a representation R :

$$Z_{1-loop}^{hyper}(\sigma, m, R) = \prod_{\rho \in R} S_3 \left(i(\rho(\sigma) + m) + \frac{E}{2} \right)^{-1},$$

where $E = \omega_1 + \omega_2 + \omega_3$.

▶ $\mathbb{R}^4 \times S^1$ inst. partition function for any of the 3 fixed points of CP^2

$$Z_{inst}^{5d,I}(\vec{\sigma}, \vec{m}, \tau; q, t) Z_{inst}^{5d,II}(\vec{\sigma}, \vec{m}, \tau; q, t) Z_{inst}^{5d,III}(\vec{\sigma}, \vec{m}, \tau; q, t)$$

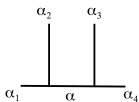
We propose

Z_{S^5} correspond to non-degenerate correlators with $\mathcal{V}ir_{qt} \otimes \mathcal{V}ir_{qt} \otimes \mathcal{V}ir_{qt}$ symmetry and S -pairing 3-point function.

We propose

Z_{S^5} correspond to non-degenerate correlators with $\mathcal{V}ir_{qt} \otimes \mathcal{V}ir_{qt} \otimes \mathcal{V}ir_{qt}$ symmetry and S -pairing 3-point function.

Example:

$$Z_{S^5}^{SQCD} = \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle_S = \int d\alpha$$


- ▶ 5d instantons vs $\mathcal{V}ir_{qt}$ non-degenerate conformal blocks:

[Awata-Yamada],[Mironov-Morozov-Shakirov-Smirnov]

$$Z_{inst}^{5d, SCQCD} = \mathcal{F}_{\alpha_1 \alpha_2 \alpha \alpha_3 \alpha_4}^{qt}$$

- ▶ 1-loop vs 3-point function:

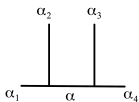
$$\mathcal{Z}_{1\text{-loop}}^{\text{vect}}(\sigma) \prod_{i=1}^4 \mathcal{Z}_{1\text{-loop}}^{\text{hyper}}(\sigma, m_i, F) = C_S(\alpha_1, \alpha_2, \alpha) C_S(E - \alpha, \alpha_3, \alpha_4)$$

with dictionary: $\alpha = i\sigma + \frac{E}{2}$, $\alpha_1 + \alpha_2 = im_1 + E$, $\alpha_1 - \alpha_2 = im_2$
 $\alpha_3 + \alpha_4 = im_3 + E$, $\alpha_3 - \alpha_4 = im_4$.

We propose

Z_{S^5} correspond to non-degenerate correlators with $\mathcal{V}ir_{qt} \otimes \mathcal{V}ir_{qt} \otimes \mathcal{V}ir_{qt}$ symmetry and S -pairing 3-point function.

Example:

$$Z_{S^5}^{SQCD} = \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle_S = \int d\alpha$$


- ▶ 5d instantons vs $\mathcal{V}ir_{qt}$ non-degenerate conformal blocks:

[Awata-Yamada], [Mironov-Morozov-Shakirov-Smirnov]

$$Z_{inst}^{5d, SCQCD} = \mathcal{F}_{\alpha_1 \alpha_2 \alpha \alpha_3 \alpha_4}^{qt}$$

- ▶ 1-loop vs 3-point function:

$$\mathcal{Z}_{1\text{-loop}}^{\text{vect}}(\sigma) \prod_{i=1}^4 \mathcal{Z}_{1\text{-loop}}^{\text{hyper}}(\sigma, m_i, F) = C_S(\alpha_1, \alpha_2, \alpha) C_S(E - \alpha, \alpha_3, \alpha_4)$$

with dictionary: $\alpha = i\sigma + \frac{E}{2}$, $\alpha_1 + \alpha_2 = im_1 + E$, $\alpha_1 - \alpha_2 = im_2$
 $\alpha_3 + \alpha_4 = im_3 + E$, $\alpha_3 - \alpha_4 = im_4$.

S_b^3 is a codimension 2 defect \rightarrow degenerate correlator

Degeneration check:

When one of the momenta degenerates, $\alpha_2 \rightarrow \omega_3/2$ then:

$$Z_{inst}^{5d,I}(\vec{\sigma}, \vec{m}, \tau; q, t) = \sum_{Y_1, Y_2} (\dots) \rightarrow \sum_{0, 1^n} (\dots) = Z_V$$

$$Z_{inst}^{5d,II}(\vec{\sigma}, \vec{m}, \tau; q, t) = \sum_{W_1, W_2} (\dots) \rightarrow \sum_{0, n} (\dots) = \tilde{Z}_V$$

$$Z_{inst}^{5d,III}(\vec{\sigma}, \vec{m}, \tau; q, t) = \sum_{X_1, X_2} (\dots) \rightarrow \sum_{0, 0} (\dots) = 1$$

so

$$Z_{inst}^{5d,I}(\vec{\sigma}, \vec{m}, \tau; q, t) Z_{inst}^{5d,II}(\vec{\sigma}, \vec{m}, \tau; q, t) Z_{inst}^{5d,III}(\vec{\sigma}, \vec{m}, \tau; q, t) \rightarrow \left\| Z_V \right\|_S^2$$

and

$$Z_{S^5}^{SQCD} \rightarrow Z_S^{SQED}$$

and similarly for permutations of $\omega_1, \omega_2, \omega_3$.

Conclusions

- ▶ evidence for a CFT-like structure underlying 5d and 3d partition functions

Conclusions

- ▶ evidence for a CFT-like structure underlying 5d and 3d partition functions

- ▶ use q -CFT to study gauge theory

Conclusions

- ▶ evidence for a CFT-like structure underlying 5d and 3d partition functions
- ▶ use q -CFT to study gauge theory
- ▶ use gauge theory to study q -CFT

$\mathcal{V}ir_{q,t}$ and 5d instanton counting

Remember Gaiotto States $|\Delta, \Lambda\rangle$:

$$L_1|\Delta, \Lambda\rangle = \Lambda^2|\Delta, \Lambda\rangle, \quad L_n|\Delta, \Lambda\rangle = 0, \quad n \geq 2$$

they satisfy:

$$\langle \Delta\Lambda | \Delta\Lambda \rangle = Z_{inst}^{4d}[SU(2)]$$

Awata and Yamada showed that deformed Gaiotto states:

$$T_1|G\rangle = \Lambda^2|G\rangle \quad T_n|G\rangle = 0, \quad n \geq 2$$

satisfy:

$$\langle G | G \rangle = Z_{inst}^{5d}[SU(2)]$$