(de)Tails of Toda CFT

Filippo Passerini
Humboldt University, Berlin

Universidad de Santiago de Compostela

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Based on: 1012.1352 Nadav Drukker and F.P.

Introduction

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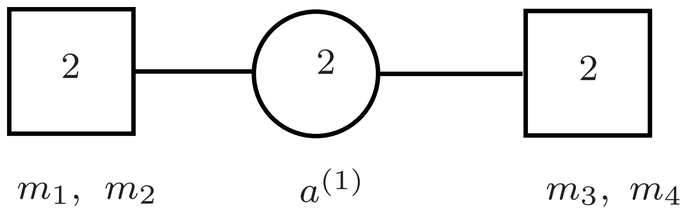
Simplest Example: $\mathcal{N}=2$ SCYM on S^4 with gauge group $SU(2)$



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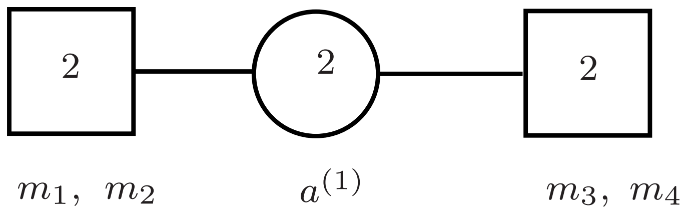

AGT:

$$Z_{S^4} = \langle V_{\beta_1} V_{\beta_2} V_{\beta_3} V_{\beta_4} \rangle_{\text{Liouville}}$$

The partition function is equivalent to a 4-point correlation function in Liouville CFT

Gauge Theory:

The partition function of $\mathcal{N}=2$ SCYM on S^4 results [Pestun]

$$Z_{S^4} = \int da^{(1)}\ Z_{\text{cl}}\ Z_{\text{1-loop}}\ Z_{\text{inst}}$$

- where
  - $Z_{\text{cl}} = e^{-\frac{4\pi^2 r^2}{g^2}\text{Tr}\,(a^{(1)})^2}$ is the classical contribution
  - $Z_{\text{1-loop}} = Z_{\text{1-loop}}\,(a,m)$ is the perturbative contribution
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The theory is conformal invariant: the Hilbert space is generated by Virasoro primaries given by

$$V_\alpha(z, \bar{z}) = e^{2\alpha\phi(z,\bar{z})}$$

- α is the primary momentum
- $\Delta(\alpha) = \alpha(Q - \alpha)$ is the conformal dimension $(Q = b + 1/b)$

Conformal Bootstrap Approach: any correlator can be expressed in terms of 3-point functions and conformal blocks

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For our basic example

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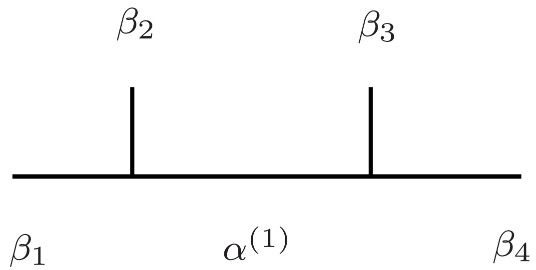
$$= \int d\alpha^{(1)} \langle\beta_1|V_{\beta_2}|\alpha^{(1)}\rangle\langle\alpha^{(1)}|V_{\beta_3}|\beta_4\rangle \mathcal{F}_{\alpha,\beta}(z)\bar{\mathcal{F}}_{\alpha,\beta}(\bar{z})\,|z|^{2(\Delta_\alpha - \Delta_{\beta_3} - \Delta_{\beta_4})}$$

and this specific decomposition is schematized by the diagram



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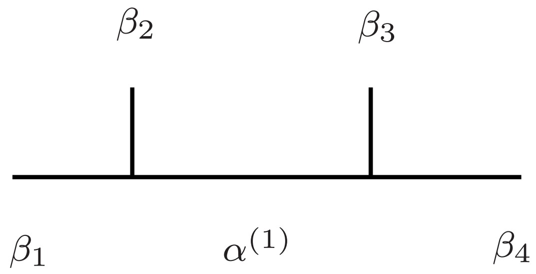
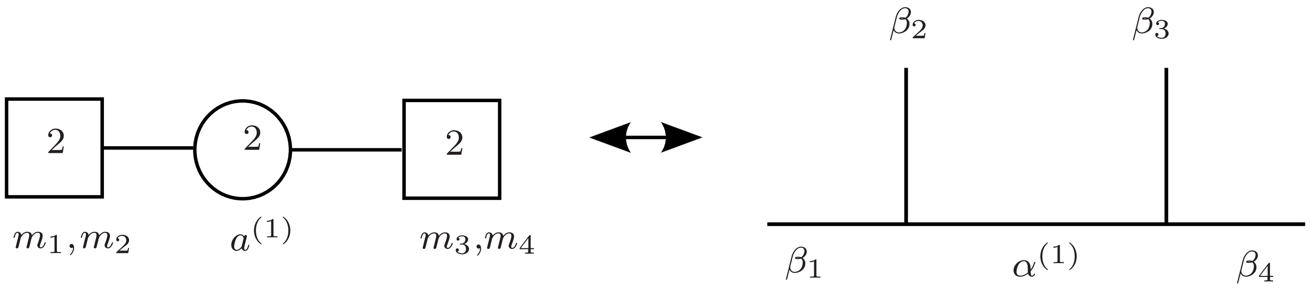


- Modular invariance: considering a different decomposition of the correlation function the result does not change

AGT



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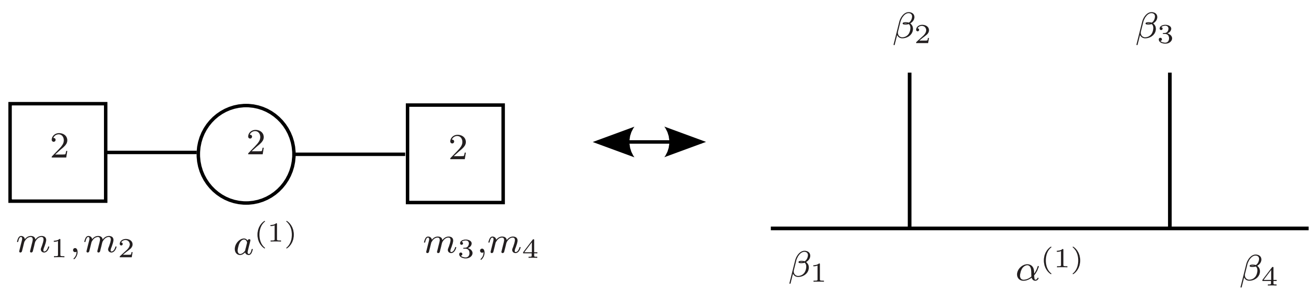

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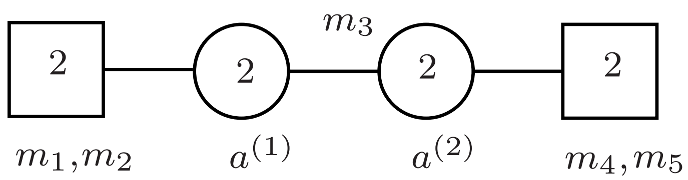
- Dictionary:
  - Coulomb branch parameters $a^{(1)}$ ⇔ primaries in the internal channels $\alpha^{(1)}$
  - Hypermultiplets masses m ⇔ primaries in the external channels β
  - $Z_{\text{1-loop}}$ ⇔ product of 3-point functions (DOZZ formulas)
  - $Z_{\rm Nek}$ ⇔ conformal block $\mathcal{F}_{\alpha,\beta}$
  - $q = e^{2\pi i \tau}$ ⇔ z

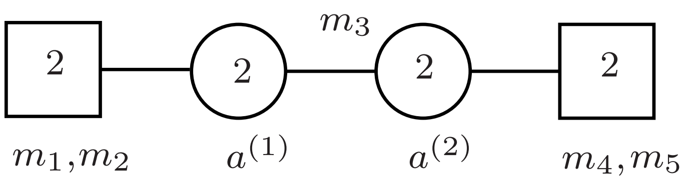
Outline

- Introduction
- The problem
- Quiver Tails
- Semi-degenerate representations of Toda CFT
- The proposal
- Proof
- Conclusion

Second example:



Second example:



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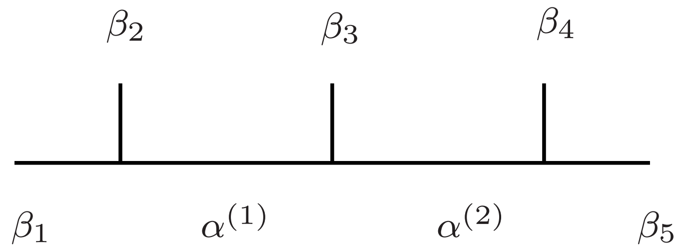

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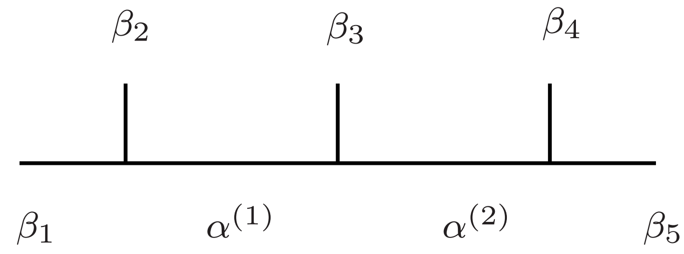

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- The dictionary works as before. In this example there are two vector multiplets:

  – Coulomb branch parameters $a^{(1)}$,$a^{(2)}$ $\Leftrightarrow$ primaries in the internal channels $\alpha^{(1)}$,$\alpha^{(2)}$

Higher Rank AGT: The partition function of conformal $\mathcal{N} = 2$ theories on S^4 with gauge group $SU(N)^k$ is equivalent to a correlation function in A_{N-1} Toda CFT [Wyllard][Kanno, Matsuo, Shiba]

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A_{N-1} Toda theory:

It is described by a 2D Lagrangian $S_{A_{N-1}} = S_{A_{N-1}}(\phi, b)$

- $\phi = \sum_{k=1}^{N-1} \varphi_k e_k$ where e_k is a simple root of A_{N-1} algebra
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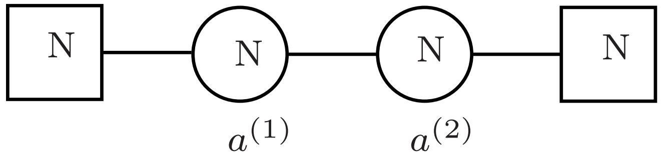
The $\mathcal{W}_N$ primaries are given by $V_\alpha \sim e^{\langle \alpha, \phi \rangle}$

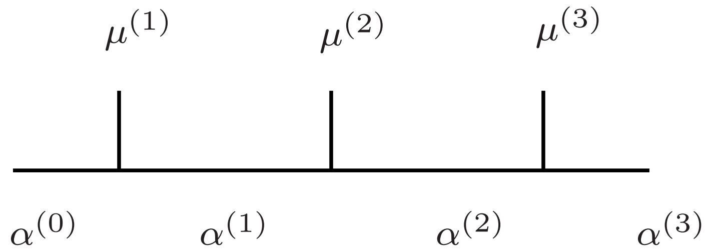
- $\langle \cdot, \cdot \rangle$ is the scalar product in the root space
- α is a vector in the root space of the A_{N-1} algebra
- the conformal dimension is given by $\Delta(\alpha) = \frac{1}{2}\langle \alpha, 2Q\rho - \alpha \rangle$

For instance:



It results: $$Z_{S^4} = \langle V_{\alpha^{(0)}} V_{\mu^{(1)}} V_{\mu^{(2)}} V_{\mu^{(3)}} V_{\alpha^{(3)}} \rangle_{A_{N-1}\ \text{Toda}}$$

where the correlation function is decomposed as

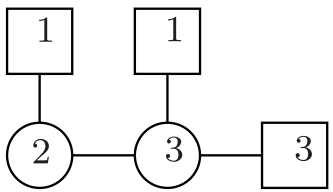


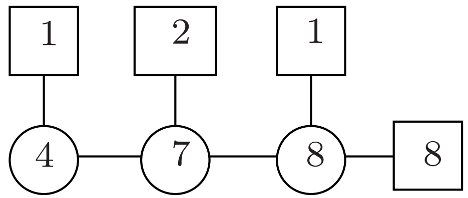
- The primary states in the internal channels $\alpha^{(1)}$, $\alpha^{(2)}$ depend each one on $N-1$ parameters. This is the dimension of the Coulomb branch of $SU(N)$ gauge group.

The problem

- There are also conformal gauge theories where the gauge group is $\prod_i SU(N_i)$ with different values of N_i.

  For instance:





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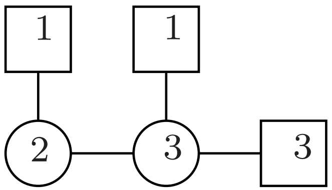

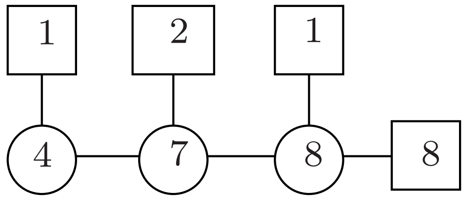

- How the partition function of this class of theories is reproduced in Toda CFT ?

Quiver Tails

- A $\mathcal{N} = 2$ $SU(N)$ gauge theory is conformal when it couples to $N_F = 2N$ matter fields in the fundamental and anti-fundamental representation

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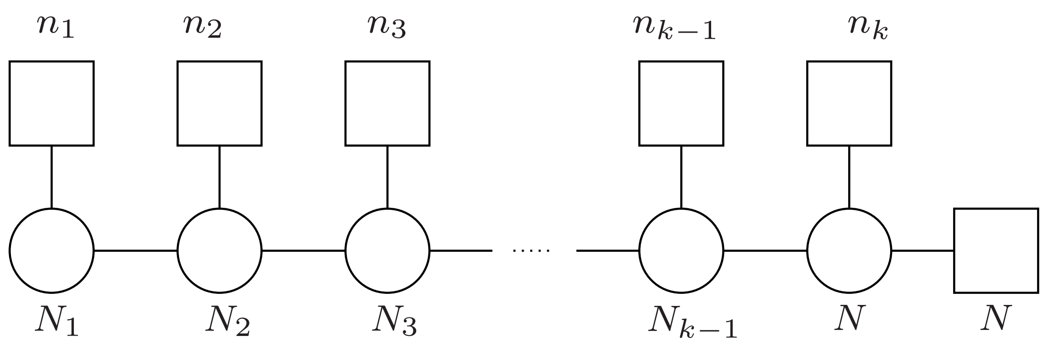
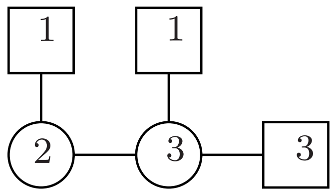
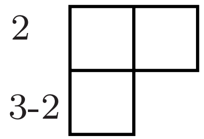
- It is possible to end a linear quiver with a finite series of of gauge groups of decreasing rank: a quiver tail [Gaiotto,Witten]



  - the theory is conformal when $n_i = 2N_i - N_{i+1} - N_{i-1} \geq 0$
  - the tail is characterized by the series $N_1 < N_2 < \ldots < N_k = N$.
    This information can be encoded in a Young diagram with N boxes with the r^{th} row of length $N_r - N_{r-1}$

- For instance





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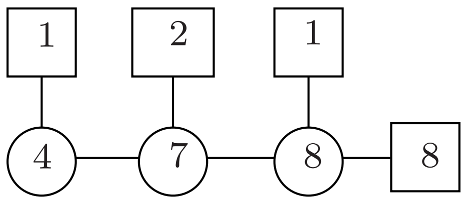

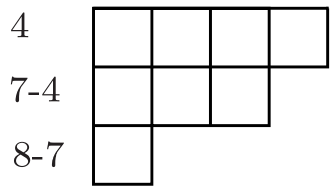

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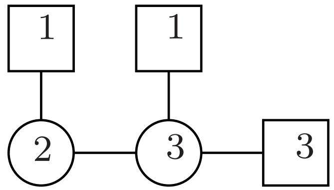

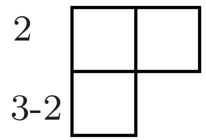

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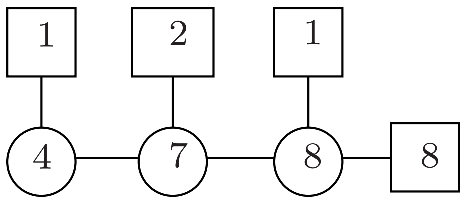

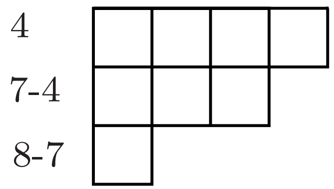

- Look for states in A_{N-1} Toda CFT characterized by a Young diagram with N boxes.

Semi-degenerate representations of Toda CFT

- A primary state of A_{N-1} Toda CFT is given by $V_\alpha \sim e^{\langle \alpha, \phi \rangle}$
  - ϕ is the Toda field defined on the root space of A_{N-1}
  - $\alpha = Q\rho + \gamma$ with $Q = b + \frac{1}{b}$ and ρ is the Weyl vector of A_{N-1}
  - in the orthonormal basis $\alpha = Q(-\frac{N+1}{2}, \dots, \frac{N+1}{2}) + (\gamma_1, \dots, \gamma_N)$
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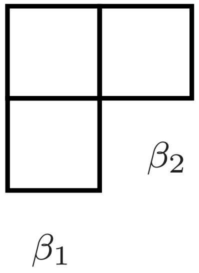
- when γ is imaginary, the descendants form an irreducible representation of $\mathcal{W}_N$ algebra
- Semi-degenerate states have descendants that are null vectors
- Physical semi-degenerate states have all the null states at level 1 [Kanno, Matsuo, Shiba, Tachikawa]
  - γ has real components
  - α is invariant under a subgroup of the permutation group S_N
  - subgroups of S_N are described by partition of N, i.e.Young diagram with N boxes

- For instance, the degenerate state of A_2 Toda



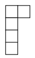
- [2,1] diagram, is invariant under $S_2 \times S_1 \subset S_3$

- The momentum is given by

$$\begin{aligned}\alpha &= Q\rho + (\beta_2,\ \beta_1 + \delta_{2,1},\ \beta_1 + \delta_{2,2}) \\ &= (\beta_2 + Q/2,\ \beta_1 - Q/2,\ \beta_1 - Q/2)\end{aligned}$$

- $\delta_{n,j} = (2j - n - 1)Q/2$

The proposal

- Claim: The partition function of a quiver gauge theory with a tail, can be expressed as a correlation function in Toda CFT [Kanno, Matsuo, Shiba, Tachikawa][Kanno, Matsuo, Shiba][Drukker,FP]
  - $SU(N)$ is maximum rank group $\Leftrightarrow$ A_{N-1} Toda CFT
  - 1 insertion is a semi-degenerate state with the same Young diagram of the tail
  - $k+1$ simple insertions $\mu^{(l)}$ 

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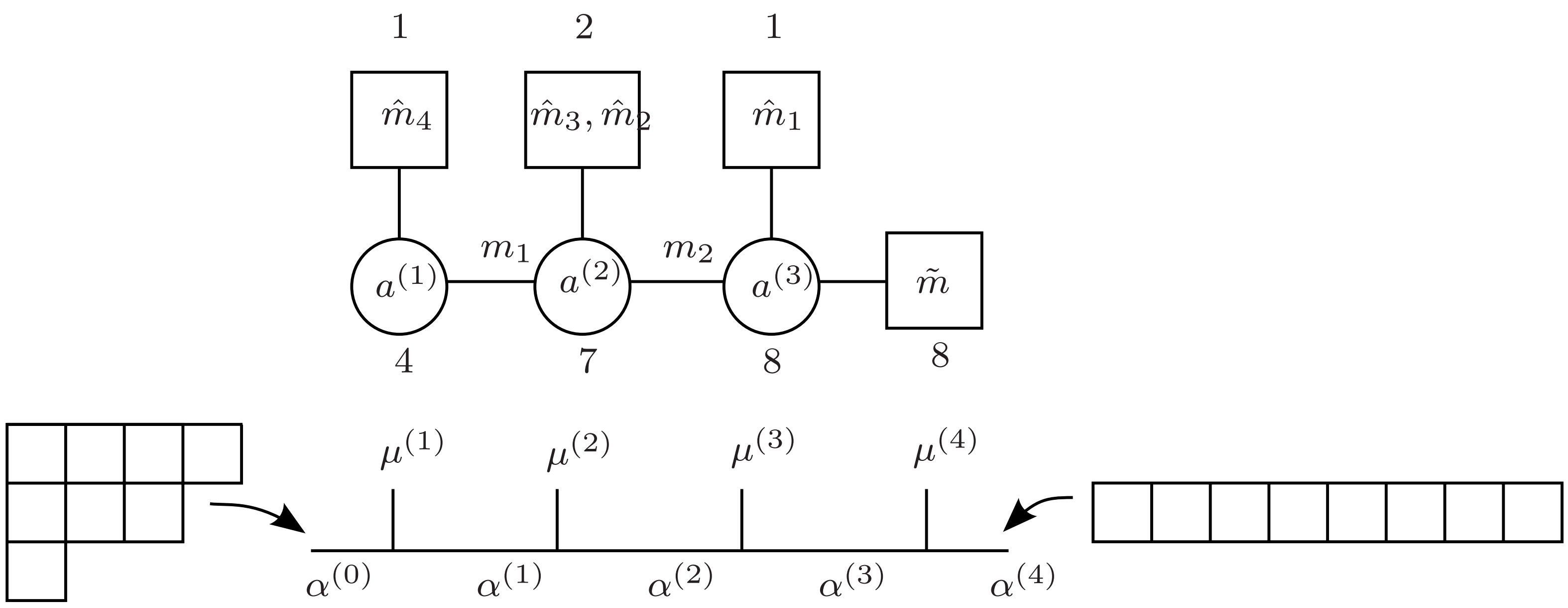

- How are the masses m_l, $\hat{m}_l$ encoded in the states $\mu^{(l)}$, $\alpha^{(0)}$, $\alpha^{(4)}$
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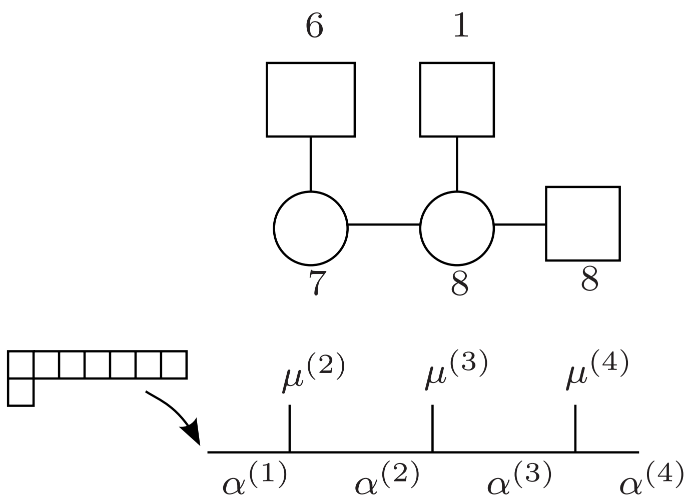

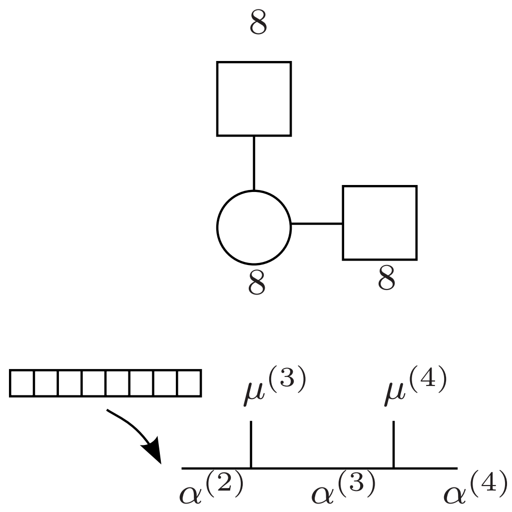

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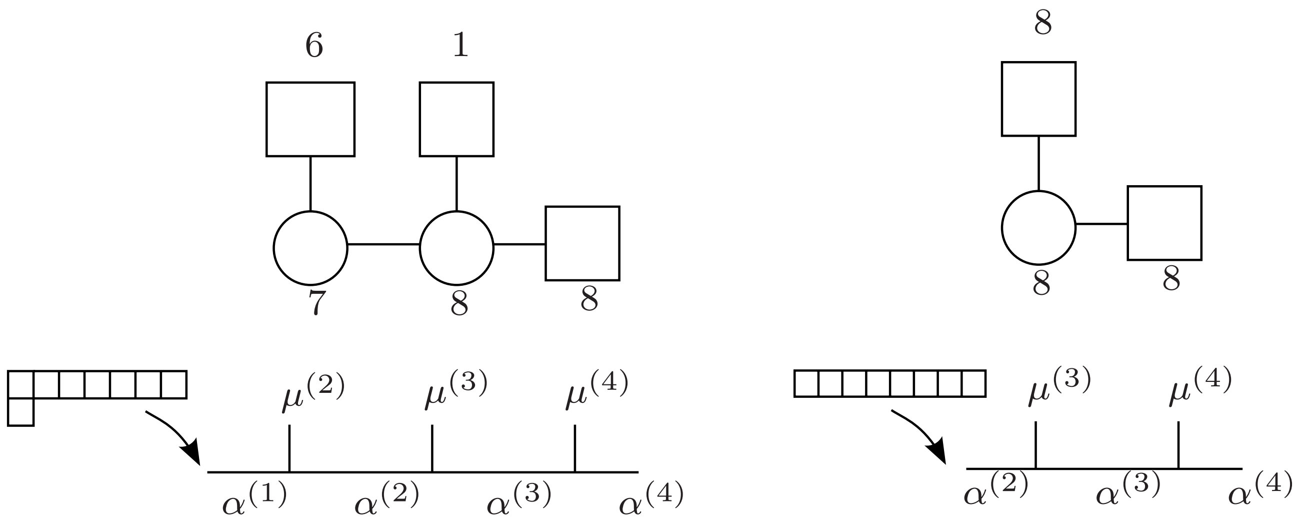
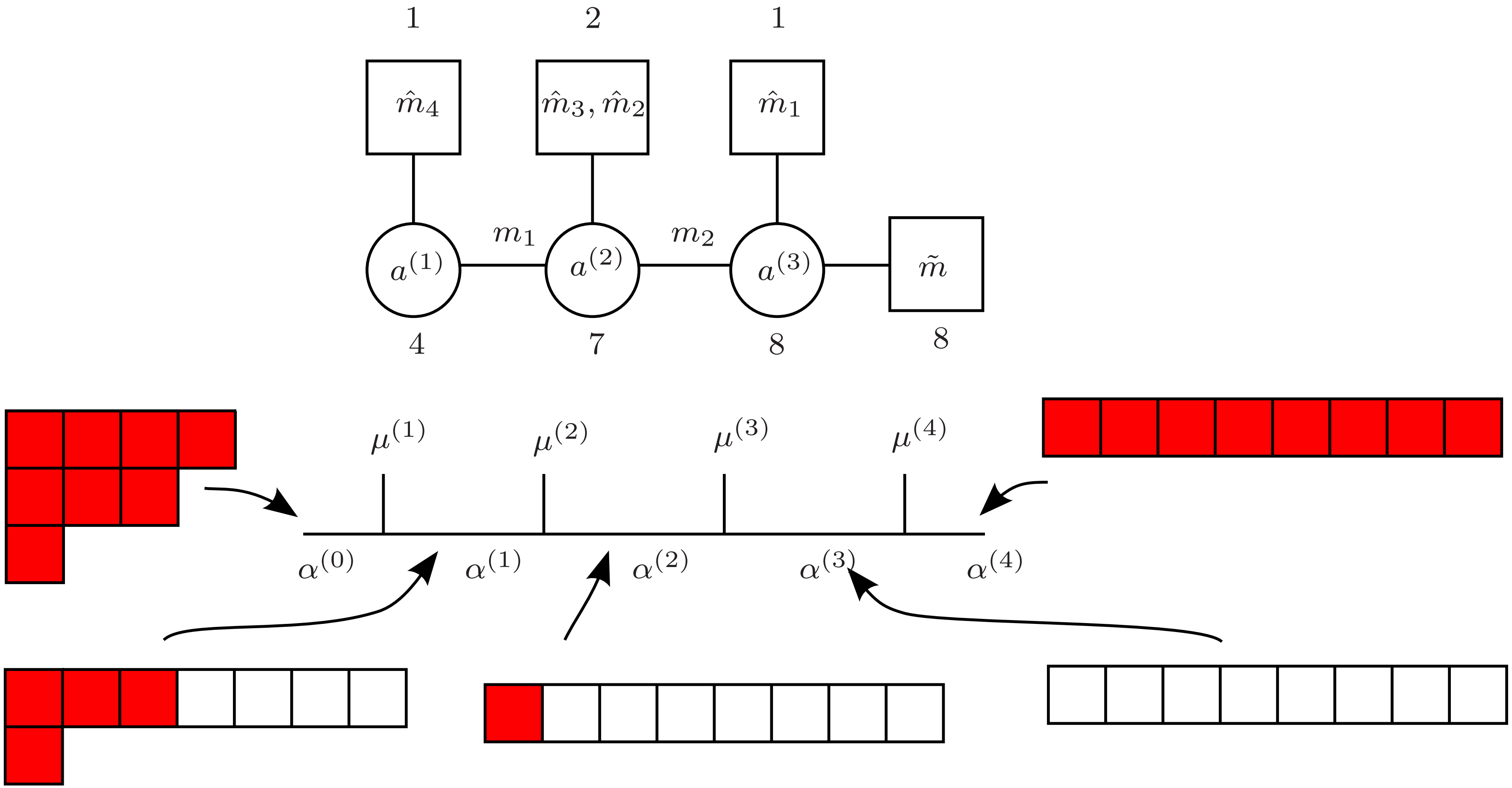


- $\alpha^{(1)}$ has 7 parameters: 3 are related to the masses, 4 to the Coulomb branch
- $\alpha^{(2)}$ has 8 parameters: 1 is related to the masses, 7 to the Coulomb branch

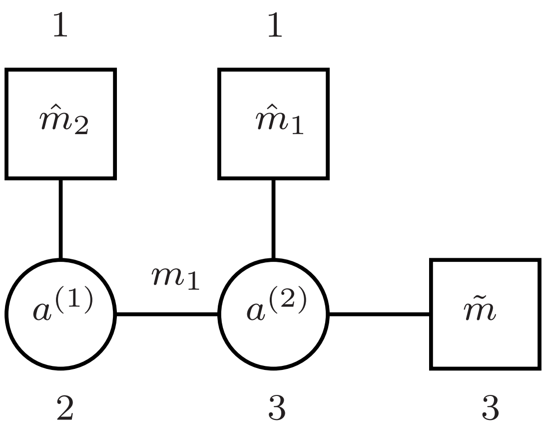
To summarize:

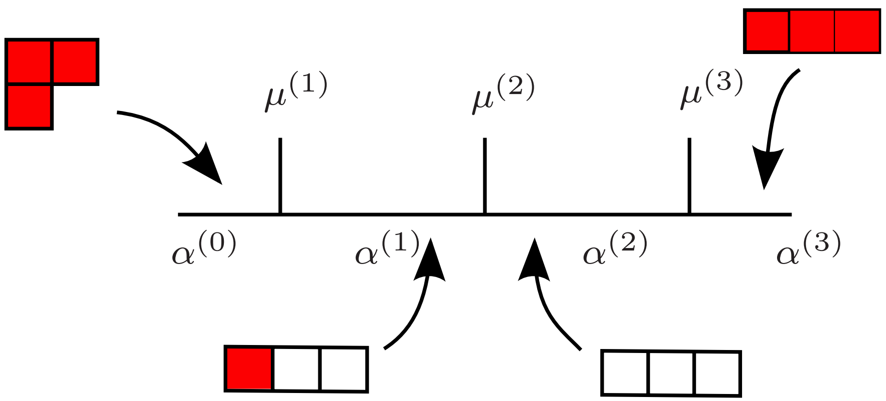


- parameters related to red boxes are fixed by mass parameters
- parameters related to white boxes are fixed by Coulomb branch parameters

Proof

Start from the $SU(2) \times SU(3)$ quiver:





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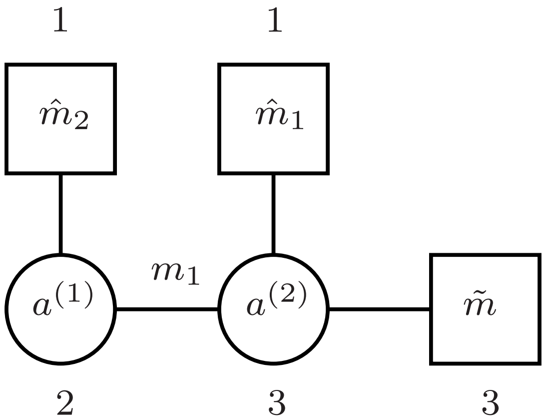

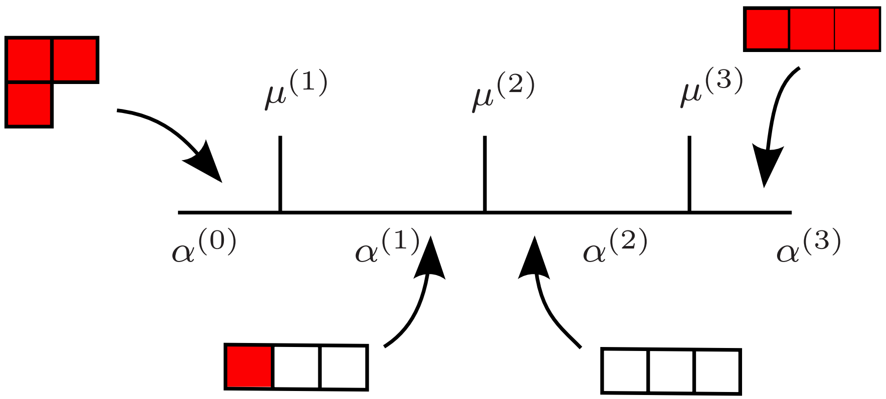

- Consider the 3-point correlation function $\langle \alpha^{(0)} | V_{\mu^{(1)}}(z) | \alpha^{(1)} \rangle$
- $\alpha^{(0)}$ and $\mu^{(1)}$ are semi-degenerate states. In particular

$$\langle \alpha^{(0)} | \left(W_1 - \frac{3w_0}{2\Delta_0} L_1 \right) \sim 0 \quad \text{where} \quad \langle \alpha^{(0)} | L_0 = \Delta_0 \langle \alpha^{(0)} | \,, \qquad \langle \alpha^{(0)} | W_0 = w_0 \langle \alpha^{(0)} |$$

- Inserting the degeneracy condition for $\alpha^{(0)}$ in the 3-point function, and permuting through $\mu^{(1)}$, produce the Ward identity

$$\left(w_1 + \frac{3}{2}\left(\frac{w_\mu}{\Delta_\mu} - \frac{w_0}{\Delta_0}\right)(\Delta_1 - \Delta_0 - \Delta_\mu) - w_0 + w_\mu\right)\left\langle \alpha^{(0)} \middle| V_{\mu^{(1)}}(z) \middle| \alpha^{(1)} \right\rangle = 0$$

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$$\begin{aligned}
\alpha^{(0)} =& Q\rho + (-2\beta^{(0)}, \beta^{(0)} - Q/2, \beta^{(0)} + Q/2) \\
\mu^{(1)} =& Q\rho + (-2\mu_1, \mu_1 - Q/2, \mu_1 + Q/2) \\
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the equation becomes $\prod_{i=1}^{3}\left(\langle \alpha^{(1)} - Q, h_i \rangle + \mu_1 - \beta^{(0)}\right) = 0$

- It results that the 3-point function is different to zero only when

$$\boxed{\beta^{(1)} = \beta^{(0)} - \mu_1}$$

Fusion rules of Toda CFT gives correct internal states as expected from gauge theory !

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The instanton corrections to the gauge theory partition function are encoded by the conformal blocks of A_2 Toda

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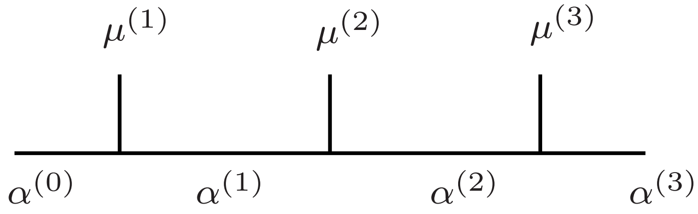

$$\mathcal{F}(q^{(1)}, q^{(2)}) = \sum_{n_1,n_2} \mathcal{F}_{n_1,n_2} (q^{(1)})^{n_1} (q^{(2)})^{n_2}$$

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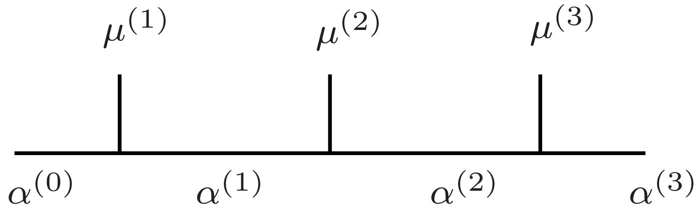

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The contribution of $SU(2)$ is associated to the $q^{(1)}$ dependent terms

$$\sum_{n_1, 0} \mathcal{F}_{n_1, 0} (q^{(1)})^{n_1} = \sum_{\mathbf{x}, \mathbf{x}'} (q^{(1)})^{|\mathbf{x}|} \frac{\langle \alpha^{(0)} | V_{\mu^{(1)}} | \mathbf{x}; \alpha^{(1)} \rangle}{\langle \alpha^{(0)} | V_{\mu^{(1)}} | \alpha^{(1)} \rangle} X^{-1}_{\mathbf{x};\mathbf{x}'}(\alpha^{(1)}) \frac{\langle \mathbf{x}'; \alpha^{(1)} | V_{\mu^{(2)}} | \alpha^{(2)} \rangle}{\langle \alpha^{(1)} | V_{\mu^{(2)}} | \alpha^{(2)} \rangle}$$

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After imposing the constraint $\boxed{\beta^{(1)} = \beta^{(0)} - \mu_1}$ it results

$$Z_{\text{inst}}(q^{(1)}) = 1 + q^{(1)} \frac{2\hat{m}_2 \prod_{i=1}^{3}(a_i^{(2)} + m^{(2)})}{Q^2 - 4a_1^{(1)}} + \mathcal{O}((q^{(1)})^2)$$

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Using the dictionary this result reproduces the contribution of the $SU(2)$ single instanton !

Let's now consider the fusion rules for semi-degenerates of A_{N-1} Toda (for any N)

- Take the 3-point function for a simple puncture and 2 non-degenerate states

$$C_{FL}\big(\alpha, (\tfrac{Q}{2} - \mu)N\omega_{N-1}, \alpha'\big) = \left[\pi\bar{\mu}\gamma(b^2)b^{2-2b^2}\right]^{\langle 2\vec{Q}-\alpha-\alpha',\rho\rangle/b}$$
$$\times \frac{(\Upsilon(b))^{N-1}\,\Upsilon\big(N(\tfrac{Q}{2}-\mu)\big)\prod_{e>0}\Upsilon\big(\langle\vec{Q}-\alpha,e\rangle\big)\Upsilon\big(\langle\vec{Q}-\alpha',e\rangle\big)}{\prod_{ij}\Upsilon\big(\tfrac{Q}{2}-\mu+\langle\alpha-\vec{Q},h_i\rangle+\langle\alpha'-\vec{Q},h_j\rangle\big)}$$

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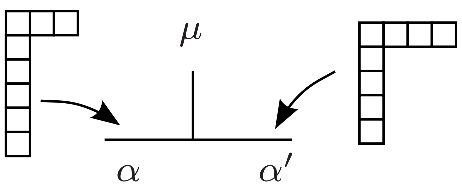
- Considering α and α' semi-degenerate states, and requiring that the formula develop a pole of the expected order, produce the right fusion rules
- Example: α and α' are "hook states" with diagrams $[n,1,\dots,1]$ and $[n',1,\dots,1]$



$$\alpha = Q\rho + \Big(-\frac{n}{N-n}\beta+\gamma_1,\cdots,-\frac{n}{N-n}\beta+\gamma_{N-n},\beta+\delta_{n,1},\cdots,\beta+\delta_{n,n}\Big)$$

Specializing to the “hook” states we find

$$C_{FL}\big(2\vec{Q}-\alpha,(\tfrac{Q}{2}-\mu)N\omega_1,\alpha'\big)=\Big[\pi\bar{\mu}\gamma(b^2)b^{2-2b^2}\Big]^{\langle\alpha-\alpha',\rho\rangle/b}\big(\Upsilon(b)\big)^{N-1}\Upsilon\big(N(\tfrac{Q}{2}-\mu)\big)$$

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- Imposing $\boxed{\beta'=\beta-\mu}$ the order of the zero becomes $\frac{(n-n')^2-n-n'}{2}$
- Requiring a pole of order n', one obtain $\boxed{n'=n-1}$

Conclusion

- The fusion rules for semi-degenerate states show that states in the internal channels are the ones expected from the gauge theory analysis
- The space of states in any internal channel has the dimension of the Coulomb branch of the related gauge multiplet
- The residues of the FL formula provide the 3-point function for 2 semi-degenerate states and a simple puncture. This 3-point functions reproduces the 1-loop part of the gauge theory partition function. This provides the precise dictionary that relates gauge theories with tails and Toda CFT
- For the $SU(2) \times SU(3)$ quiver, we have shown that the W_3 conformal blocks reproduce the $SU(2)$ instantons, once we take into account the constraints that relate the primaries
- Some future directions:
  - Non-local operators in quiver tails
  - Generalization to non-conformal theories
  - Possible different description coupling Toda CFT's with different ranks