Optimal Trading with Alpha Predictors

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Introduction

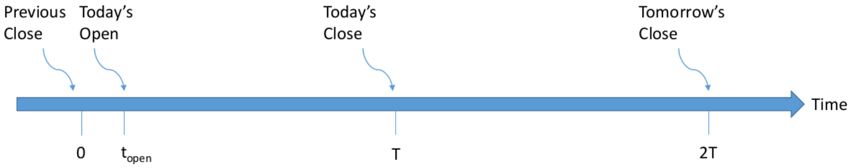
- Some alpha strategies are based on daily data.
- Others are constructed using high-frequency or intraday signals.
- Some strategies benefit from market-making with limit orders.
- How can we optimally combine these different alpha streams?
- In this work we present a practical framework inspired by the Hamilton-Jacobi-Bellman (HJB) method.
- We do not seek to find exact analytical solutions to the optimization problem, but to introduce some algorithmic “recipes” which can be used in real trading.

Introduction

- We begin by considering the case of market orders only.
- We then study how to optimally use limit orders.
- We define the problem in continuous time, but always keeping in mind the practical implementation, which is in discrete time.
- Our framework allows us to take into account linear costs and price impact.

Time Axis

- We are allowed to trade from time t_{open} until time T.
- We will optimize the integrated P&L from some time $t \in [t_{\text{open}}, T]$ until time $2T$.



Notation

- For simplicity, we will consider a single asset with price P_t. The dynamics of the price is given by

$$dP_t = \alpha_t dt + \sqrt{\nu} dW_t$$

where dt is the time scale of our trading decisions.

- Without loss of generality, we will decompose the drift into a constant $\bar{\alpha}$ and intraday component x_t with zero mean:

$$\alpha_t := \bar{\alpha} + x_t \ , \quad \mathbb{E}[x_t] = 0$$

- **We can think of $\bar{\alpha}$ as the alpha that comes from the daily predictors, while x_t comes from the intraday or HF signals.**

Notation

- At this point we leave the dynamics of x_t to be quite general:

$$dx_t = \mu(t, x_t)dt + \sqrt{\eta(t, x_t)}dZ_t$$

- However, we usually model the fast signal as a mean-reverting process with constant volatility: $dx_t = -\kappa x_t dt + \sqrt{\eta}dZ_t$.
- For later convenience, we will introduce the differential operator:

$$\hat{D}_{t,x} := \frac{\partial}{\partial t} + \mu(t, x)\frac{\partial}{\partial x} + \frac{1}{2}\eta(t, x)\frac{\partial^2}{\partial x^2}$$

Notation

- We introduce the integrated gain of the HF signal:

$$g(t, x) := \int_t^{2T} \mathbb{E}\left[x_s \middle| x_t = x\right] ds$$

  which obeys:

$$\hat{D}_{t,x} \cdot g(t, x) + x = 0$$

- The evolution of our position q_t is given by:

$$dq_t = u_t dt$$

- **The optimization problem boils down to finding the optimal rate of trading u_t.**

Optimization with Market Orders

- We begin by considering the optimal trading problem using market orders only. The objective function is given by:

$$\Omega(t,x,q) = \min_{\{u_s|s\in(t,T)\}} \mathbb{E}\left[\int_t^T C|u_s|ds + K\int_t^T u_s^2 ds - \int_t^{2T} \alpha_s q_s ds + \frac{1}{2}\lambda\nu\int_t^{2T} q_s^2 ds \,\middle|\, q_t = q, x_t = x\right]$$

- The first term is the cost of the market orders.
- The second term serves as a *control* or regulator for the size of the trades. One can also see it as coming from market impact.
- The third term is the gain coming from the alpha signals.
- The last term is a risk control.

Optimization with Market Orders

- The constant K can be interpreted as a "regulator" that controls the average trade size.
- However, it can also be interpreted as coming from market impact. In fact, we can model a general permanent impact function: $K \int u_s^p ds$.
- Another useful formulation of the problem is to assume constant and discretized trading rate, that is $u \in [-Q, 0, Q]$.

Optimization with Market Orders

- Using the Hamilton-Jacobi-Bellman principle, we can derive the PDE for the objective function:

$$\hat{D}_{t,x} \cdot \Omega(t,x,q) - (\bar{\alpha} + x)q + \frac{1}{2}\lambda\nu q^2 + \min_u \left[C|u| + Ku^2 + \frac{\partial \Omega}{\partial q} u \right] = 0$$

- We can write the optimization as a *portfolio tracking problem*. For that, we introduce the daily Markowitz portfolio:

$$\bar{q} := \frac{\bar{\alpha}}{\lambda\nu}$$

and a new function:

$$V(t,x,q) := \Omega(t,x,q) + g(t,x)q + \frac{1}{2}\lambda\nu\bar{q}^2(2T - t)$$

Tracking Markowitz

- To gain some intuition of the meaning of V it is useful to write it as

$$V(t,x,q) = \min_{\{u_s|s\in(t,T)\}} \mathbb{E}\left[\int_t^T (C|u_s| - g(s,x_s)u_s)\,ds + K\int_t^T u_s^2 ds + \frac{1}{2}\lambda\nu\int_t^{2T}(q_s-\bar{q})^2 ds \,\middle|\, q_t = q, x_t = x\right]$$

- The first term is the effective cost of the market order once we take into account the HF predictors.
- The last term is the integrated residual risk (variance) of the residual between our position and the daily Markowitz portfolio.

Tracking Markowitz

- The optimization is dual to tracking the Markowitz portfolio using the fast signals:

$$\hat{D}_{t,x} \cdot V + \frac{1}{2}\lambda\nu(q-\bar{q})^2 + \min_u \left[C|u| + Ku^2 + \left(\frac{\partial V}{\partial q} - g \right) u \right] = 0$$

with boundary condition:

$$V(T,x,q) = \frac{1}{2}\lambda\nu T(q-\bar{q})^2$$

- **The boundary condition is the residual variance around the Markowitz portfolio.**

Trading and No-Trading Zones

- The solution to the optimization problem is divided into three regions.

1. $g > C + \frac{\partial V}{\partial q}$. In this case $u > 0$, so we **buy**:

$$u = \frac{1}{2K}\left(g - C - \frac{\partial V}{\partial q}\right)$$

2. $g < -C + \frac{\partial V}{\partial q}$. In this case $u < 0$, so we **sell**:

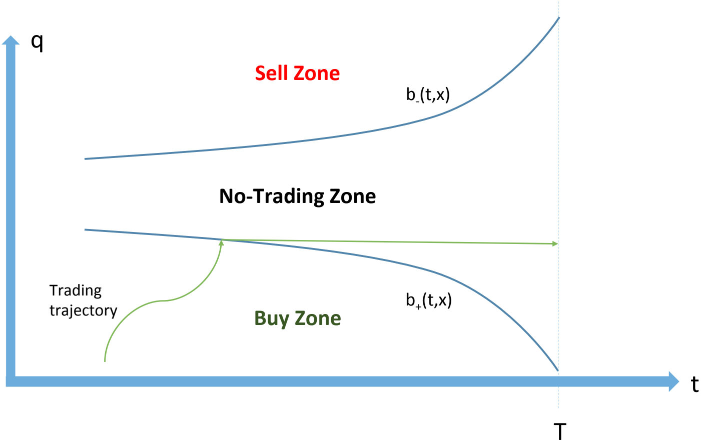
$$u = -\frac{1}{2K}\left(-g - C + \frac{\partial V}{\partial q}\right)$$

3. $-C + \frac{\partial V}{\partial q} \leq g \leq C + \frac{\partial V}{\partial q}$. In this case we **don't trade** $(u = 0)$.

- linear costs induces a "no-trading" zone when using market orders.

Trading and No-Trading Zones

- For a constant signal (e.g. $x_t = 0$), the trading zones typically look as follows:



Trading and No-Trading Zones

- We can summarize the trade rate as:

$$u = \frac{1}{2K}\left(g - C - \frac{\partial V}{\partial q}\right)_+ - \frac{1}{2K}\left(-g - C + \frac{\partial V}{\partial q}\right)_+$$

- The full equation for V can now be written in compact form:

$$\hat{D}_{t,x} \cdot V + \frac{1}{2}\lambda\nu(q - \bar{q})^2 - Ku^2 = 0$$

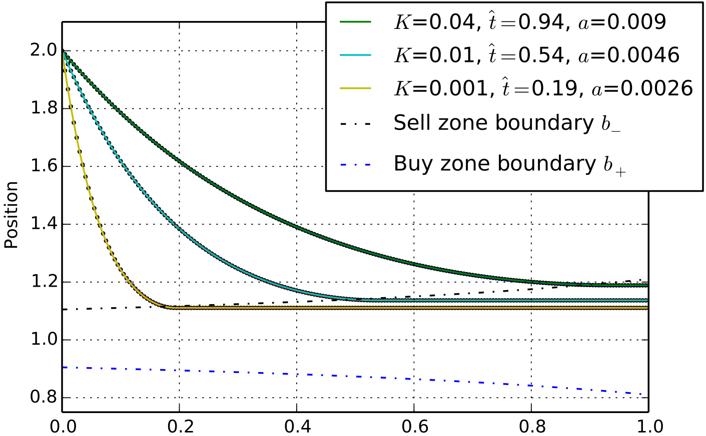
with boundary condition:

$$V(T, x, q) = \frac{1}{2}\lambda\nu T(q - \bar{q})^2$$

- Very difficult PDE to solve in general.

Zero volatility regime

- In the case of a zero volatility signal, there is no uncertainty and the optimization problem can be solved exactly using the Euler-Lagrange variational principle:



Zero volatility regime

- A general feature of the solution is that as $K \to 0$ we go instantaneously to one of the boundaries of the NT zone (if we are out).
- As $K \to \infty$ we trade slowly towards the boundary.
- In the case of a constant or exponential signal, once we enter the no-trading (NT) zone, we stay there until the end of the day.
- In the stochastic case, however, the boundaries of this region depend on x_t and hence one can go in and out of the region many times.
- **How can we determine the boundaries of the no-trading zone for the stochastic case?**

Trading and No-Trading Zones

- The HJB equation is very difficult to solve in general.
- For practical purposes, we propose using the solution that comes from ignoring the source term $-Ku^2$:

$$V \approx \frac{1}{2}\lambda\nu(2T - t)(q - \bar{q})^2$$

- Can we justify this approximation?

Trading and No-Trading Zones

- In the limit $K \to \infty$, the trading rate goes to zero as $u \sim 1/K$ and we're justified to use the approximation. In fact, one can define a systematic expansion in $1/K$.
- The other, more practical limit is when $K \to 0$. In this case, we trade towards the boundary of the NT zone instantaneously. One can argue that in this case u diverges, but in practice, the trade size must be finite and hence

  $$dq = udt = \text{finite}$$

  Moreover, in practice dt is finite (e.g. one second). Therefore, u remains finite as K goes to zero and we can justify our approximation $Ku^2 \sim 0$.

Recipe 1

- This is, of course, not a formal mathematical proof but only an argument for our first recipe:

$$\begin{aligned} \frac{\partial V}{\partial q} &= \lambda\nu(2T-t)(q-\bar{q}) \\ u &= \frac{1}{2K}\left(g - C - \frac{\partial V}{\partial q}\right)_+ - \frac{1}{2K}\left(-g - C + \frac{\partial V}{\partial q}\right)_+ \end{aligned}$$

- Within our approximation, the boundaries of the NT zone are given by:

$$\boxed{b_\pm(t,x) = \bar{q} + \frac{1}{\lambda\nu(2T-t)}\left(g(t,x) \mp C\right)}$$

Implementation

- One can take $dt = 1$ to be the basic trading decision scale, and u our trade size.
- If we are small and we can ignore impact, we can take $K \to 0$ and trade instantaneously towards one of the boundaries.
- The Lagrange multiplier λ can be written as:

$$\lambda = \frac{\text{Annualized Sharpe Ratio of Daily target}}{\text{Annualized Volatility of Daily Target}}$$

- We can take $\bar{q}$ to be the ideal positions that come from a daily back test.

Implementation

- We can model high-frequency predictors as mean-reverting processes.
- In practice, it is useful to decompose our intraday alpha in terms of a z-score or signal ϵ_t with $\mathbb{E}[\epsilon_t] = 0$ and $\mathrm{Var}[\epsilon_t] = 1$.
- We can write:

$$\begin{aligned} x_t &:= \beta\sqrt{\nu}\epsilon_t \\ d\epsilon &= -\kappa\epsilon dt + \sqrt{2\kappa}dZ \end{aligned}$$

Implementation

- The constant β can be conveniently written in terms of the annual Sharpe ratio of the ideal HF position $\tilde{q}_t = \epsilon_t/\sqrt{\nu}$:

$$\beta = \frac{\text{Annualized Sharpe Ratio of HF Signal}}{\sqrt{252T}}$$

- The gain is then:

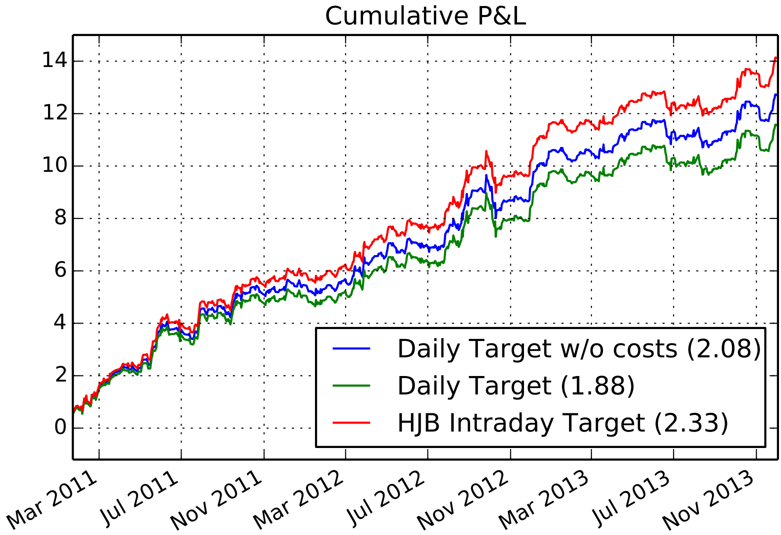
$$g(t,\epsilon) = \frac{\beta\sqrt{\nu}\epsilon}{\kappa}\left(1 - e^{-\kappa(2T-t)}\right) \approx \frac{\beta\sqrt{\nu}\epsilon}{\kappa}$$

where the last approximation is valid if the time scale of the predictor is much smaller than one day.

Simulation

- In the following figure, we show a Monte Carlo simulation of our algorithm.
- We trade directly to the boundary of the NT zone in one shot.
- Our decision time scale is $dt = 1$ minute.
- The intraday signal has a mean-reversion of 30 mins.
- The daily signal is constant during the day, but varies from day to day with a mean-reversion time scale of 10 days and a annual Sharpe of 2.
- Other relevant parameters are $\nu = 0.01$, $C = 0.01$, $\lambda = 37.4$, $\beta = 1$ (Sharpe $= 16.5$).

Simulation



Cumulative P&L
14
12
10
8
6
4
2
0
Mar 2011
Jul 2011
Nov 2011
Mar 2012
Jul 2012
Nov 2012
Mar 2013
Jul 2013
Nov 2013
Daily Target w/o costs (2.08)
Daily Target (1.88)
HJB Intraday Target (2.33)

Limit Orders

- We can use the same HJB framework with limit orders.
- Our position will now evolve according to:

$$dq_t = (m_t^+ - m_t^- + \mathbf{1}_t^+ l_t^+ - \mathbf{1}_t^- l_t^-)dt \ , \quad m_t^\pm, l_t^\pm \geq 0$$

where $m_t^\pm$ and $l_t^\pm$ are the sizes of the market and limit orders, and $\mathbf{1}_t^\pm = 1$ if the limit order is executed in the interval dt and zero otherwise.

- Hence, we assume that our limit orders are small enough to be filled in one shot (we ignore partial fills).

Assumptions

- We also assume that we place limit orders at the top of the book and hence do not optimize for their price. However, this optimization can easily be implemented within our framework.
- At the end of the interval dt all pending limit orders are canceled.
- We do not allow a limit and market order with the opposite sign as they will trivially execute against each other. Hence, one must choose one or the other. However, we are allowed to send both a limit buy and sell order (market making).
- We will ignore price impact and simply cap the size of out trades:

$$\begin{aligned} m^{\pm} &\in [0, Q] \\ l^{\pm} &\in [0, Q] \\ m^{\pm} + l^{\pm} &\in [0, Q] \end{aligned}$$

Fill Probabilities

- Let P^+, P^- be the conditional probability that a limit buy and sell order will be filled within the next time step dt.
- So far we have modeled the price in our decision interval dt as a diffusion process. However, in real life, there are price movements inside dt which are discrete (tick-by-tick). These are the movements which must be predicted by $P^{\pm}$.
- **Hence for limit orders, we must have a very short term predictor y_t which might be different from the longer-term intraday alpha x_t.**

$$P^{\pm}(t, x, y) := \mathbb{E}[\mathbf{1}_t^{\pm} | x_t = x, y_t = y]$$

The HJB Equation

- The HJB Equation takes the following form:

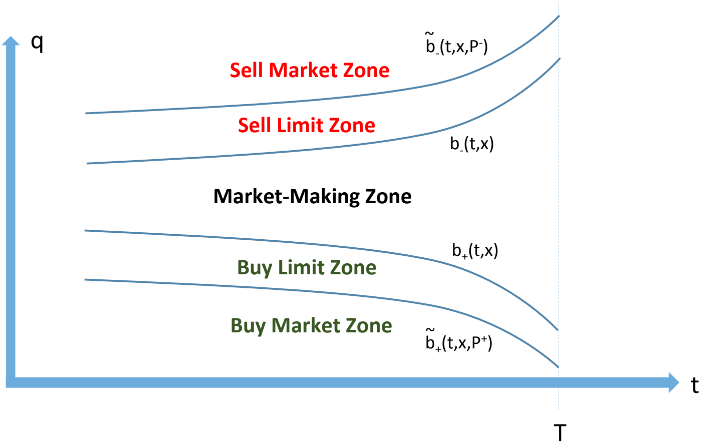
$$\begin{aligned} 0 &= \hat{D}_{t,x,y} \cdot V + \frac{1}{2}\lambda\nu(q-\bar{q})^2 \\ &+ \min_{m^{\pm},l^{\pm}} \left[m^+ \left(C + \frac{\partial V}{\partial q} - g \right) - P^+ l^+ \left(C - \frac{\partial V}{\partial q} + g \right) \right. \\ &\left. + m^- \left(C - \frac{\partial V}{\partial q} + g \right) - P^- l^- \left(C + \frac{\partial V}{\partial q} - g \right) \right] \end{aligned}$$

- Note that, in general, the dependency of the probabilities and the gain are of the following form:

$$P^{\pm} = P^{\pm}(t,x,y) \ , \quad g = g(t,x)$$

Optimization

- Our trading decisions are based on the optimization over $m^{\pm}, l^{\pm}$ in the HJB equation.
- As in the case of market orders, we find different trading regions.
- There are in total five regions:
  1. $g > C\frac{1+P^+}{1-P^+} + \frac{\partial V}{\partial q}$ we send a **buy market** order.
  2. $C + \frac{\partial V}{\partial q} < g < C\frac{1+P^+}{1-P^+} + \frac{\partial V}{\partial q}$ we send a **buy limit** order.
  3. $-C + \frac{\partial V}{\partial q} \leq g \leq C + \frac{\partial V}{\partial q}$ we **send both a buy and a sell limit order**.
  4. $-C\frac{1+P^-}{1-P^-} + \frac{\partial V}{\partial q} < g < -C + \frac{\partial V}{\partial q}$ we send a **sell limit** order.
  5. $g < -C\frac{1+P^-}{1-P^-} + \frac{\partial V}{\partial q}$ we send a **sell market** order.



- The no-trading region is now replaced by a market-making region.

Recipe 2

- What are the optimal value of Q?
- As in the case of market orders, in the absence of impact, we would like to trade as much as possible. However, we should not cross trading regions because this will lead to more unnecessary back-and-forth trading in the future.
- **Hence, it is optimal to trade to the boundary of our current trading region.**

Recipe 2

- As in the case of market orders, a full solution of HJB seems hopeless.
- We approximate V to be the value function of the no-trading zone:

$$V(t,x,q) \approx \frac{1}{2}\lambda\nu(2T-t)(q-\bar{q})^2$$

- We can write the boundaries explicitely:

$$\begin{aligned} b_{\pm}(t,x) &= \bar{q} + \frac{1}{\lambda\nu(2T-t)}\left(g(t,x) \mp C\right) \\ \tilde{b}_{\pm}(t,x,P^{\pm}) &= \bar{q} + \frac{1}{\lambda\nu(2T-t)}\left(g(t,x) \mp C\frac{1+P^{\pm}}{1-P^{\pm}}\right) \end{aligned}$$

Implementation

- For simplicity, we will assume that the spread is constant (one tick) and hence limit orders are only executed if there is a mid price move in the right direction.
- In our simplistic case with constant spread, we find that it is suboptimal to send both buys and sells limit orders at the same time (market making). Instead, it is optimal to not trade in this region.

Implementation

- For our simulations, in order to capture the very short-term behavior of the price relevant to limit orders, we decompose our intraday alpha as a “slow” (ϵ_t) and “fast” ($\tilde{\epsilon}_t$) predictors:

$$\begin{aligned} dP_t &= (\bar{\alpha} + x_t)dt + \sqrt{\nu}dW_t \\ x_t &= \sqrt{\nu}\left(\beta\epsilon_t + \tilde{\beta}\tilde{\epsilon}_t\right) \\ d\epsilon_t &= -\kappa\epsilon_t dt + \sqrt{2\kappa}dZ_t \\ d\tilde{\epsilon}_t &= -\tilde{\kappa}\tilde{\epsilon}_t dt + \sqrt{2\tilde{\kappa}}d\tilde{Z}_t \end{aligned}$$

Implementation

- The mean reversion scale of $\tilde{\epsilon}_t$ will be of the order of dt:

$$\tilde{\kappa} \sim 1/dt$$

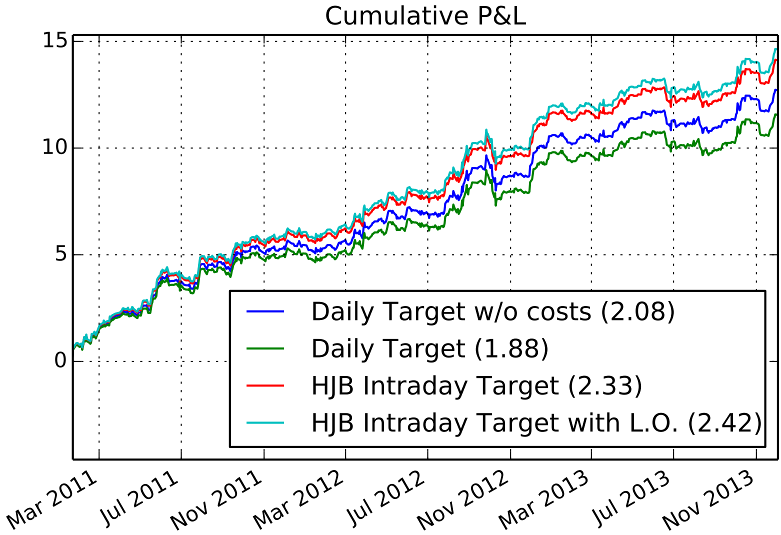
- In this case, the fill probabilities will be functions of $\tilde{\epsilon}$ only, while the long-term integrated gain will be only a function of ϵ_t:

$$P^{\pm} \approx P^{\pm}(\tilde{\epsilon}) \ , \quad g(t, \epsilon) \approx \frac{\beta\sqrt{\nu}\epsilon}{\kappa}$$

Simulation

- In the next figure we show the result of a Monte Carlo simulation with our simple limit/market order algorithm.
- All parameters are the same as in our previous simulation, with the exception of the new signal $\tilde{\epsilon}_t$.
- We take this signal to have a mean-reversion time scale of 1 minute (dt) and $\tilde{\beta} = 13$.
- As in the previous simulation, we trade instantaneously towards the boundary of our zone.
- We find a mild improvement from the use of limit orders.

Simulation


Cumulative P&L
15
10
5
0
Daily Target w/o costs (2.08)
Daily Target (1.88)
HJB Intraday Target (2.33)
HJB Intraday Target with L.O. (2.42)
Mar 2011
Jul 2011
Nov 2011
Mar 2012
Jul 2012
Nov 2012
Mar 2013
Jul 2013
Nov 2013

Conclusions

- We have developed a general framework to think about optimal algorithmic trading using Hamilton-Jacobi-Bellman theory.
- Even tough the HJB equations cannot be solved exactly, we have presented various analytic "recipes" or algorithms which are inspired on general features of the exact solution.
- Our framework has allowed us to unify, not only daily and intraday alpha signals, but also market and limit orders.

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