(de)Tails of Toda CFT

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Based on: 1012.1352 Nadav Drukker and F.P.

Introduction

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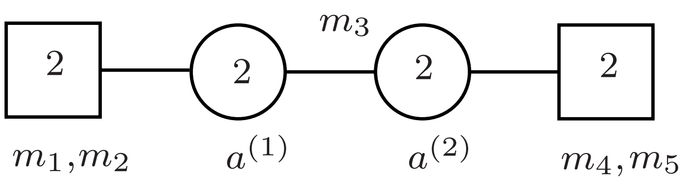

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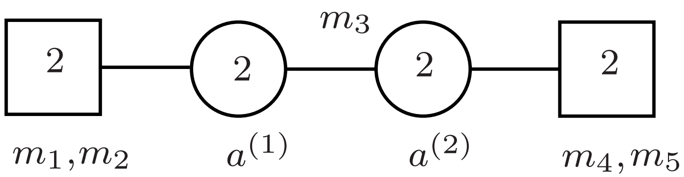


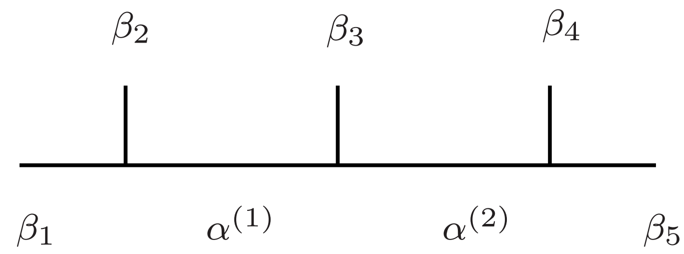
The partition function on S^4 results [Pestun]

$$Z_{S^4} = \int da^{(1)} da^{(2)} \; Z_{\text{Nek}} \, \bar{Z}_{\text{Nek}}$$

- where $Z_{\text{Nek}} = Z_{\text{cl}} \, Z_{\text{1-loop}} \, Z_{\text{inst}}$
- the partition function localizes to a matrix integral in $a^{(1)}$ and $a^{(2)}$, the VEVs of the scalars in the vector multiplets that parameterize the Coulomb branch

- AGT: $$\boxed{Z_{S^4} = \langle V_{\beta_1} V_{\beta_2} V_{\beta_3} V_{\beta_4} V_{\beta_5} \rangle_{\text{Liouville}}}$$

- the correlation function is decomposed as



$$\langle V_{\beta_1} V_{\beta_2} V_{\beta_3} V_{\beta_4} V_{\beta_5} \rangle = \int d\alpha^{(1)} d\alpha^{(2)} \langle \beta_1 | V_{\beta_2} | \alpha^{(1)} \rangle \langle \alpha^{(1)} | V_{\beta_3} | \alpha^{(2)} \rangle \langle \alpha^{(2)} | V_{\beta_4} | \beta_5 \rangle \mathcal{F}_{\alpha,\beta} \bar{\mathcal{F}}_{\alpha,\beta}$$

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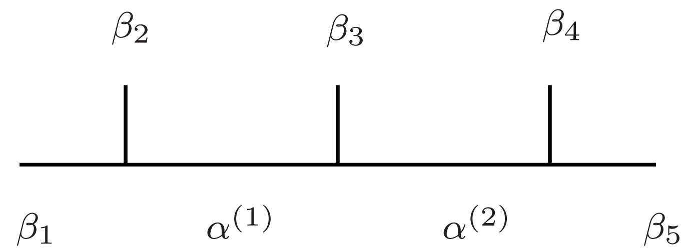

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- Dictionary:
  - Coulomb branch parameters $a^{(1)}$,$a^{(2)}$ $\Leftrightarrow$ primaries in the internal channels $\alpha^{(1)}$,$\alpha^{(2)}$
  - Hypermultiplets masses $\Leftrightarrow$ primaries in the external channels
  - $|Z_{\text{1-loop}}|^2$ $\Leftrightarrow$ product of 3-point functions (DOZZ formulas)
  - Z_{inst} $\Leftrightarrow$ conformal block $\mathcal{F}_{\alpha,\beta}$

Higher Rank AGT: The partition function of conformal $\mathcal{N} = 2$ theories on S^4 with gauge group $SU(N)^k$ is equivalent to a correlation function in A_{N-1} Toda CFT [Wyllard][Kanno, Matsuo, Shiba]

Higher Rank AGT: The partition function of conformal $\mathcal{N} = 2$ theories on S^4 with gauge group $SU(N)^k$ is equivalent to a correlation function in A_{N-1} Toda CFT [Wyllard][Kanno, Matsuo, Shiba]

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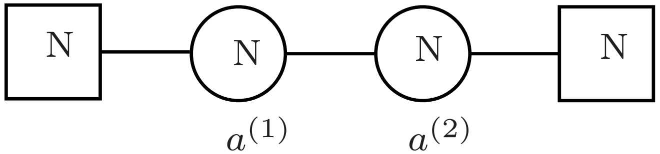
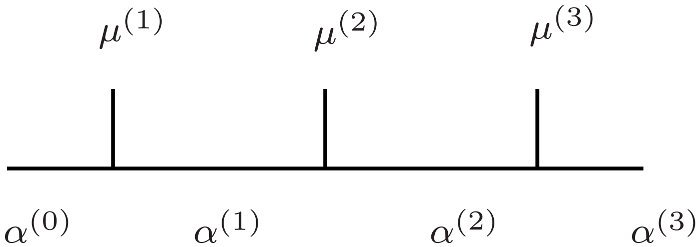


It results:

$$Z_{S^4} = \langle V_{\alpha^{(0)}} V_{\mu^{(1)}} V_{\mu^{(2)}} V_{\mu^{(3)}} V_{\alpha^{(3)}} \rangle_{A_{N-1}\ \text{Toda}}$$

where the correlation function is decomposed as



- The primary states in the internal channels $\alpha^{(1)}$, $\alpha^{(2)}$ depend each one on $N-1$ parameters. This is the dimension of the Coulomb branch of $SU(N)$ gauge group.

The problem

- There are also conformal gauge theories where the gauge group is $\prod_i SU(N_i)$ with different values of N_i.

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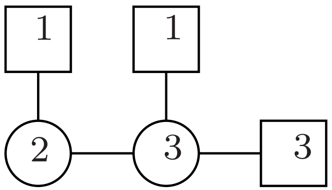

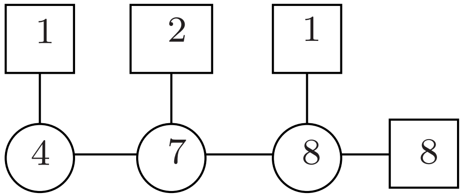

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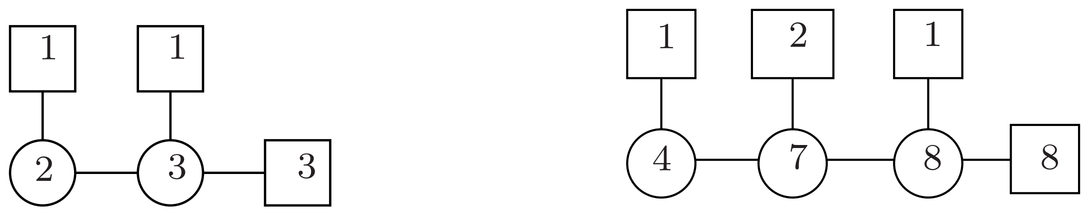


- How the partition function of this class of theories is reproduced in Toda CFT ?

Outline

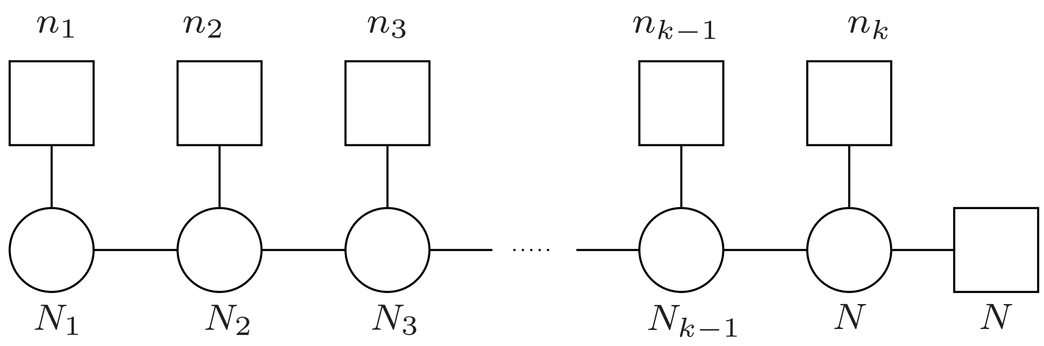
- Introduction
- The problem
- Quiver Tails
- Semi-degenerate representations of Toda CFT
- The proposal
- Proof
- Conclusion

Quiver Tails

- A $\mathcal{N} = 2$ $SU(N)$ gauge theory is conformal when it couples to $N_F = 2N$ matter fields in the fundamental and anti-fundamental representation

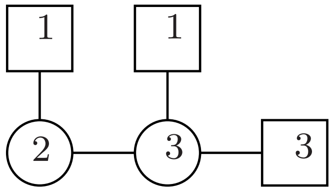
Quiver Tails

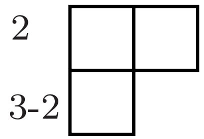
- A $\mathcal{N} = 2$ $SU(N)$ gauge theory is conformal when it couples to $N_F = 2N$ matter fields in the fundamental and anti-fundamental representation
- It is possible to end a linear quiver with a finite series of of gauge groups of decreasing rank: a quiver tail [Gaiotto,Witten]



  - the theory is conformal when $n_i = 2N_i - N_{i+1} - N_{i-1} \geq 0$
  - the tail is characterized by the series $N_1 < N_2 < \ldots < N_k = N$. This information can be encoded in a Young diagram with N boxes with the r^{th} row of length $N_r - N_{r-1}$

- For instance





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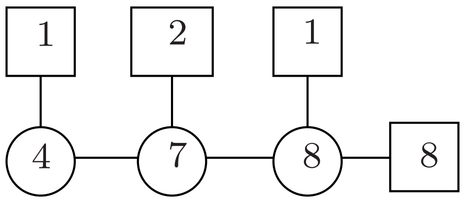

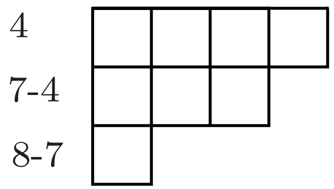

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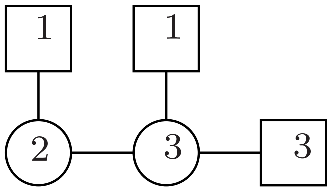

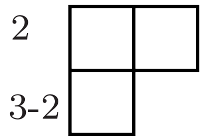

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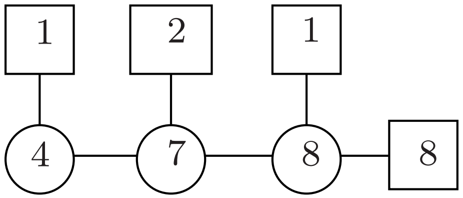

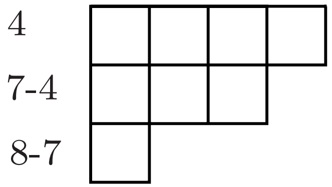

- Look for states in A_{N-1} Toda CFT characterized by a Young diagram with N boxes.

Semi-degenerate representations of Toda CFT

- A primary state of A_{N-1} Toda CFT is given by $V_\alpha \sim e^{\langle \alpha, \phi \rangle}$
  - ϕ is the Toda field defined on the root space of A_{N-1}
  - $\alpha = Q\rho + \gamma$ with $Q = b + \frac{1}{b}$ and ρ is the Weyl vector of A_{N-1}
  - in the orthonormal basis $\alpha = Q(-\frac{N+1}{2}, \dots, \frac{N+1}{2}) + (\gamma_1, \dots, \gamma_N)$
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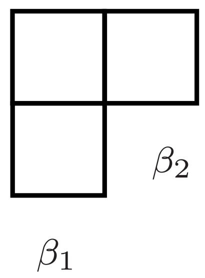
- when γ is imaginary, the descendants form an irreducible representation of W_N algebra
- Semi-degenerate states have descendants that are null vectors
- Physical semi-degenerate states have all the null states at level 1 [Kanno, Matsuo, Shiba, Tachikawa]
  - γ has real components
  - α is invariant under a subgroup of the permutation group S_N
  - subgroups of S_N are described by partition of N, i.e.Young diagram with N boxes

- For instance, the degenerate state of A_2 Toda



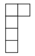
- [2,1] diagram, is invariant under $S_2 \times S_1 \subset S_3$

- The momentum is given by

$$\begin{aligned}\alpha &= Q\rho + (\beta_2,\ \beta_1 + \delta_{2,1},\ \beta_1 + \delta_{2,2}) \\ &= (\beta_2 + Q/2,\ \beta_1 - Q/2,\ \beta_1 - Q/2)\end{aligned}$$

- $\delta_{n,j} = (2j - n - 1)Q/2$

The proposal

- Claim: The partition function of a quiver gauge theory with a tail, can be expressed as a correlation function in Toda CFT [Kanno, Matsuo, Shiba, Tachikawa][Kanno, Matsuo, Shiba][Drukker,FP]
  - $SU(N)$ is maximum rank group $\Leftrightarrow$ A_{N-1} Toda CFT
  - 1 insertion is a semi-degenerate state with the same Young diagram of the tail
  - $k+1$ simple insertions $\mu^{(l)}$ 

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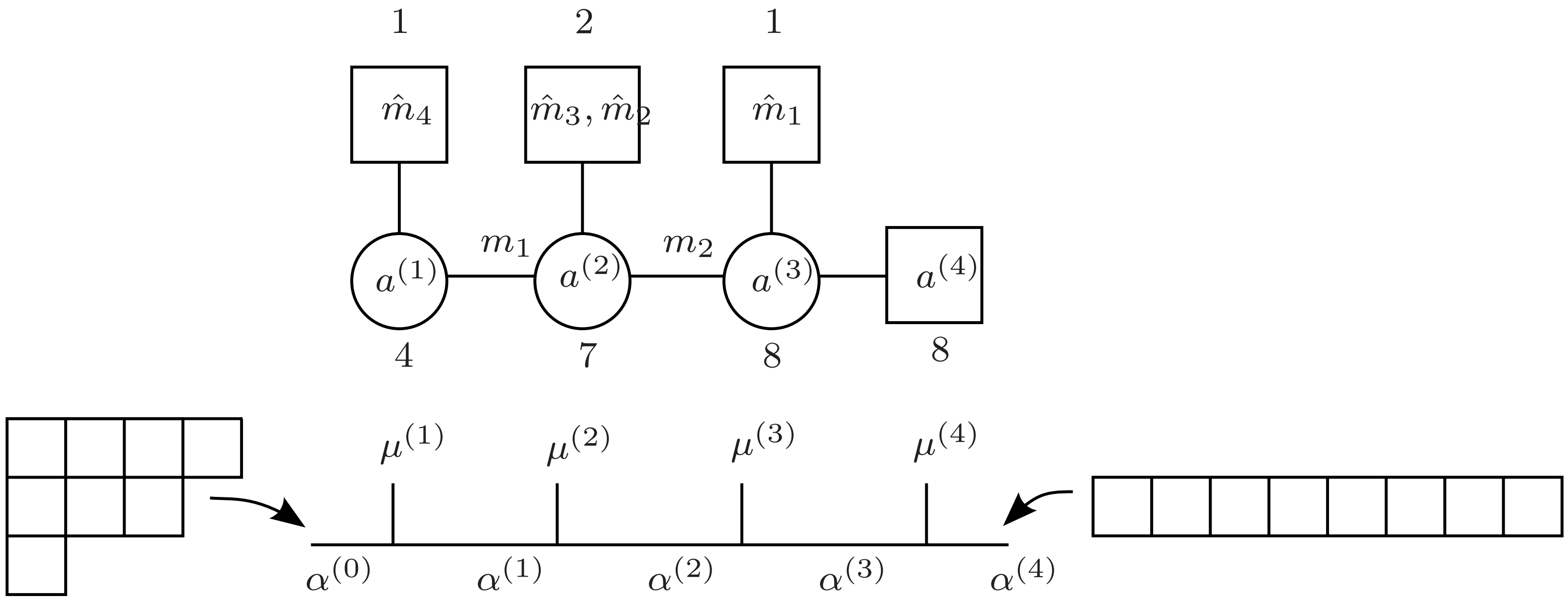

- How are the masses m_l, $\hat{m}_l$ encoded in the states $\mu^{(l)}$, $\alpha^{(0)}$, $\alpha^{(4)}$
- What are the allowed states $\alpha^{(l)}$ in the internal channels, and how are they related to the Coulomb branch parameters $a^{(l)}$

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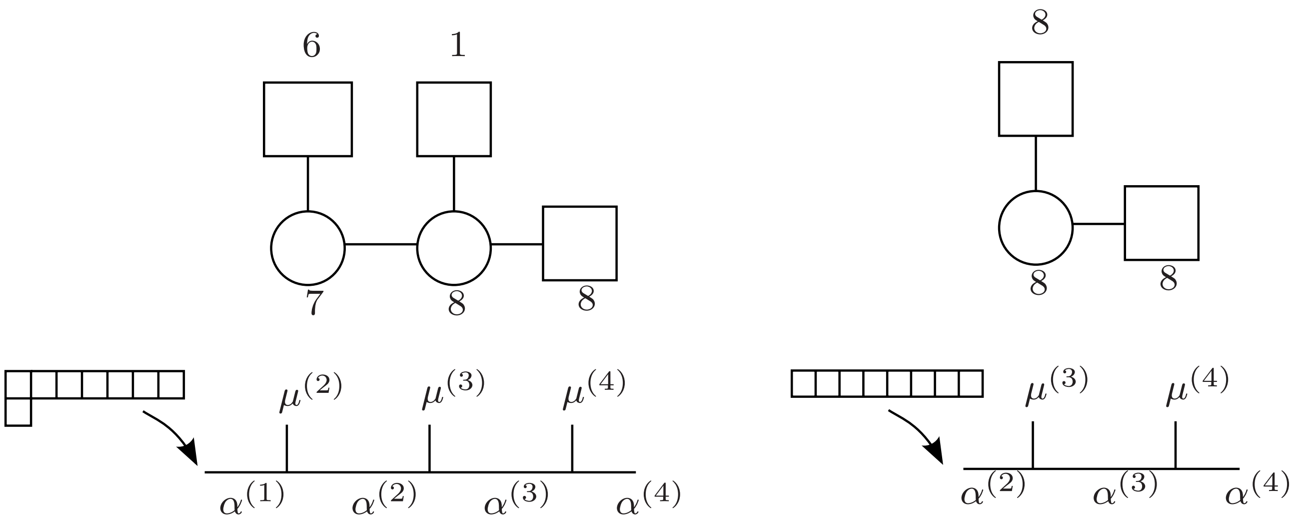

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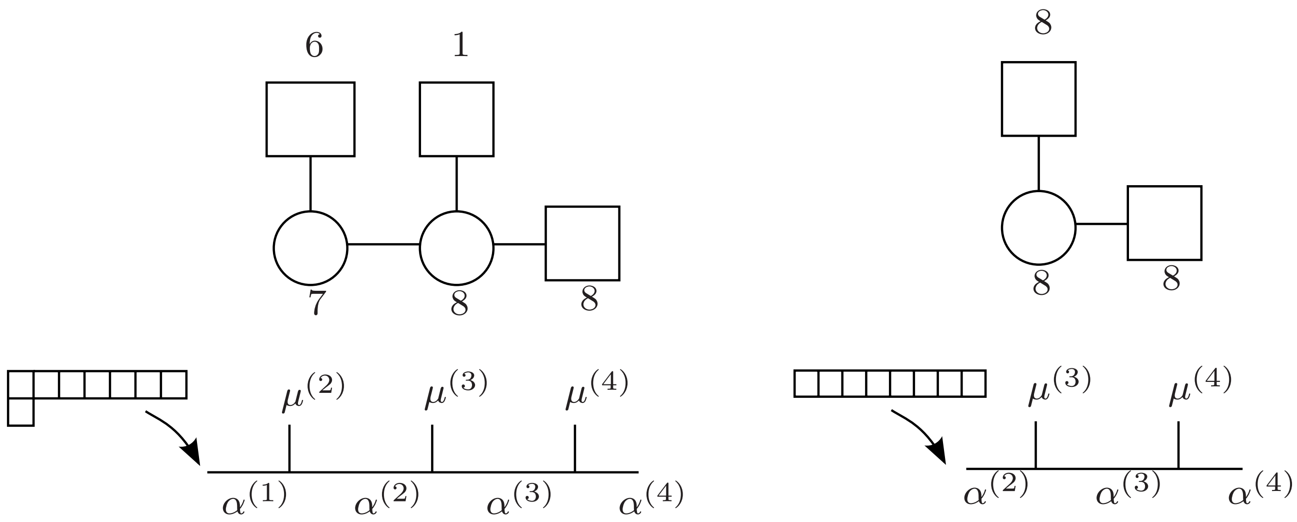
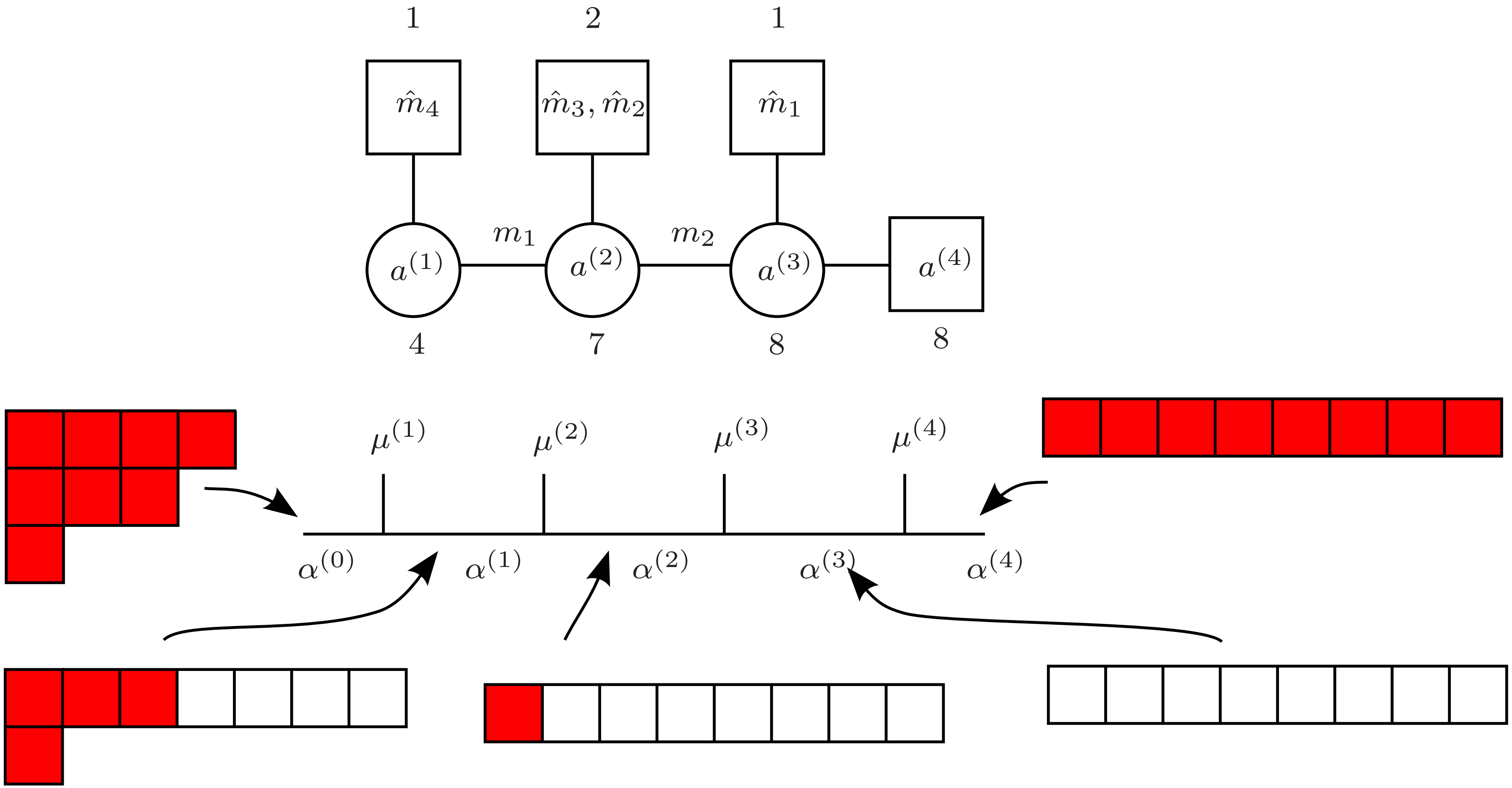


- $\alpha^{(1)}$ has 7 parameters: 3 are related to the masses, 4 to the Coulomb branch
- $\alpha^{(2)}$ has 8 parameters: 1 is related to the masses, 7 to the Coulomb branch

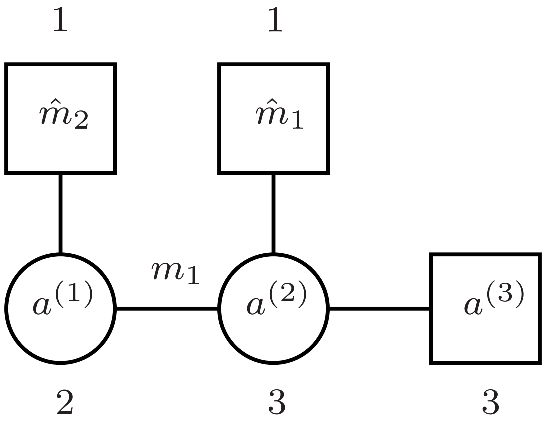
To summarize:

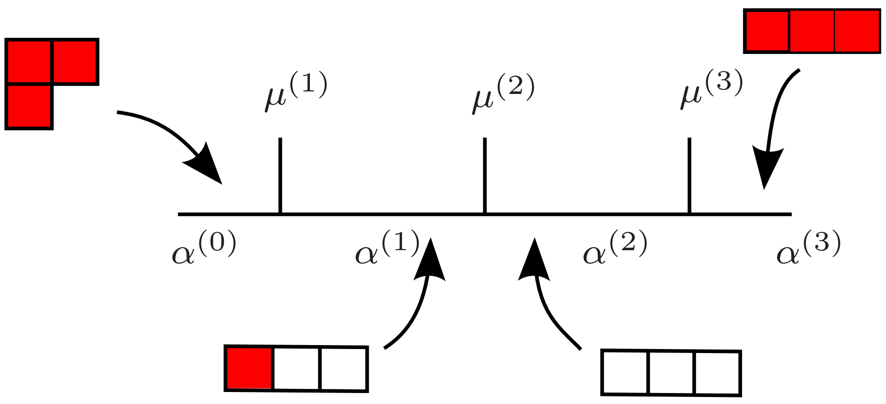


- parameters related to red boxes are fixed by mass parameters
- parameters related to white boxes are fixed by Coulomb branch parameters

Proof

Start from the $SU(2) \times SU(3)$ quiver:





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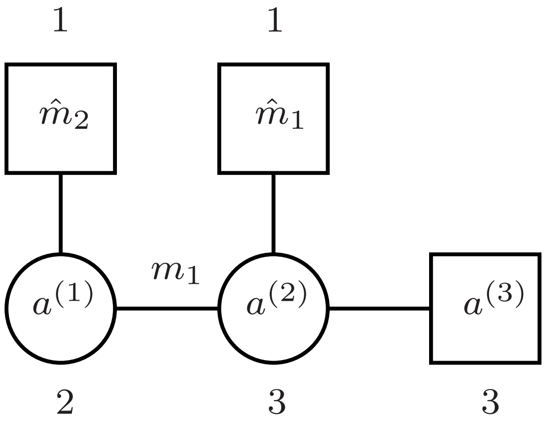

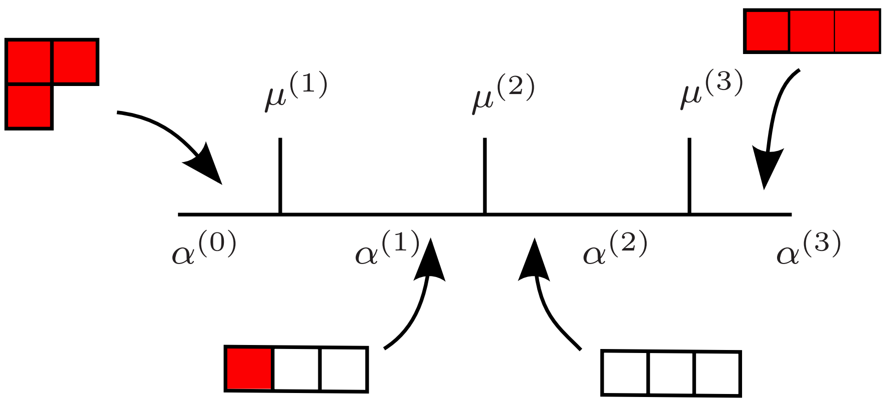

- Consider the 3-point correlation function $\langle \alpha^{(0)} | V_{\mu^{(1)}}(z) | \alpha^{(1)} \rangle$
- $\alpha^{(0)}$ and $\mu^{(1)}$ are semi-degenerate states. In particular

$$\langle \alpha^{(0)} | \left(W_1 - \frac{3w_0}{2\Delta_0} L_1 \right) \sim 0 \quad \text{where} \quad \langle \alpha^{(0)} | L_0 = \Delta_0 \langle \alpha^{(0)} | \,, \qquad \langle \alpha^{(0)} | W_0 = w_0 \langle \alpha^{(0)} |$$

- Inserting the degeneracy condition for $\alpha^{(0)}$ in the 3-point function, and permuting through $\mu^{(1)}$, produce the Ward identity

$$\left(w_1 + \frac{3}{2}\left(\frac{w_\mu}{\Delta_\mu} - \frac{w_0}{\Delta_0}\right)(\Delta_1 - \Delta_0 - \Delta_\mu) - w_0 + w_\mu\right)\left\langle \alpha^{(0)} \middle| V_{\mu^{(1)}}(z) \middle| \alpha^{(1)} \right\rangle = 0$$

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- Considering the parameterization

$$\begin{aligned}\alpha^{(0)} =& Q\rho+(-2\beta^{(0)},\beta^{(0)}-Q/2,\beta^{(0)}+Q/2)\\ \mu^{(1)} =& Q\rho+(-2\mu_1,\mu_1-Q/2,\mu_1+Q/2)\\ \alpha^{(1)} =& Q\rho+(\bar{\beta}^{(1)}+\gamma_1^{(1)},\bar{\beta}^{(1)}+\gamma_2^{(1)},\beta^{(1)})\end{aligned}$$

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- It results that the 3-point function is different to zero only when

$$\boxed{\beta^{(1)} = \beta^{(0)} - \mu_1}$$

Fusion rules of Toda CFT gives correct internal states as expected from gauge theory !

Let's now consider the fusion rules for semi-degenerates of A_{N-1} Toda (for any N)

- Take the 3-point function for a simple puncture and 2 non-degenerate states

$$C_{FL}\big(\alpha, (\tfrac{Q}{2}-\mu)N\omega_{N-1}, \alpha'\big) = \Big[\pi\bar{\mu}\gamma(b^2)b^{2-2b^2}\Big]^{\langle 2\vec{Q}-\alpha-\alpha',\rho\rangle/b}$$
$$\times \frac{\big(\Upsilon(b)\big)^{N-1}\Upsilon\big(N(\tfrac{Q}{2}-\mu)\big)\prod_{e>0}\Upsilon\big(\langle\vec{Q}-\alpha,e\rangle\big)\Upsilon\big(\langle\vec{Q}-\alpha',e\rangle\big)}{\prod_{ij}\Upsilon\big(\tfrac{Q}{2}-\mu+\langle\alpha-\vec{Q},h_i\rangle+\langle\alpha'-\vec{Q},h_j\rangle\big)}$$

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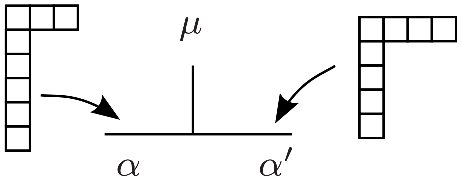
- Considering α and α' semi-degenerate states, and requiring that the formula develop a pole of the expected order, produce the right fusion rules

- Example: α and α' are "hook states" with diagrams $[n, 1, \ldots, 1]$ and $[n', 1, \ldots, 1]$



$$\alpha = Q\rho + \Big(-\frac{n}{N-n}\beta + \gamma_1, \cdots, -\frac{n}{N-n}\beta + \gamma_{N-n}, \beta + \delta_{n,1}, \cdots, \beta + \delta_{n,n}\Big)$$

Specializing to the "hook" states we find

$$C_{FL}\big(2\vec{Q}-\alpha,(\tfrac{Q}{2}-\mu)N\omega_1,\alpha'\big)=\Big[\pi\bar{\mu}\gamma(b^2)b^{2-2b^2}\Big]^{\langle\alpha-\alpha',\rho\rangle/b}\big(\Upsilon(b)\big)^{N-1}\Upsilon\big(N(\tfrac{Q}{2}-\mu)\big)$$

$$\times\frac{\prod_{i<j\leq N-n}\Upsilon\big(\gamma_i-\gamma_j\big)\prod_{i<j\leq N-n'}\Upsilon\big(\gamma'_j-\gamma'_i\big)}{\prod_{i\leq N-n}\prod_{j\leq N-n'}\Upsilon\big(\frac{Q}{2}-\mu-\frac{n}{N-n}\beta+\frac{n'}{N-n'}\beta'+\gamma_i-\gamma'_j\big)}$$

$$\times\frac{\prod_{j\leq n}\prod_{i\leq N-n}\Upsilon\big(-\frac{N}{N-n}\beta+\gamma_i-\delta_{n,j}\big)}{\prod_{i\leq N-n}\prod_{j\leq n'}\Upsilon\big(\frac{Q}{2}-\mu-\frac{n}{N-n}\beta-\beta'+\gamma_i-\delta_{n',j}\big)}$$

$$\times\frac{\prod_{i\leq N-n'}\prod_{j\leq n'}\Upsilon\big(\frac{N}{N-n'}\beta'-\gamma'_i+\delta_{n',j}\big)}{\prod_{j\leq N-n'}\prod_{i\leq n}\Upsilon\big(\frac{Q}{2}-\mu+\beta+\frac{n'}{N-n'}\beta'-\gamma'_j+\delta_{n,i}\big)}$$

$$\times\frac{\prod_{i<j\leq n}\Upsilon(\delta_{n,i}-\delta_{n,j})\prod_{i<j\leq n'}\Upsilon(\delta_{n',j}-\delta_{n',i})}{\prod_{i\leq n}\prod_{j\leq n'}\Upsilon\big(\frac{Q}{2}-\mu+\beta-\beta'+\delta_{n,i}-\delta_{n',j}\big)}$$

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$$
\begin{aligned}
C_{FL}\big(2\vec{Q}-\alpha,(\tfrac{Q}{2}-\mu)N\omega_1,\alpha'\big) &= \Big[\pi\bar{\mu}\gamma(b^2)b^{2-2b^2}\Big]^{\langle\alpha-\alpha',\rho\rangle/b}\big(\Upsilon(b)\big)^{N-1}\Upsilon\big(N(\tfrac{Q}{2}-\mu)\big) \\
&\times\frac{\prod_{i<j\le N-n}\Upsilon\big(\gamma_i-\gamma_j\big)\prod_{i<j\le N-n'}\Upsilon\big(\gamma'_j-\gamma'_i\big)}{\prod_{i\le N-n}\prod_{j\le N-n'}\Upsilon\big(\frac{Q}{2}-\mu-\frac{n}{N-n}\beta+\frac{n'}{N-n'}\beta'+\gamma_i-\gamma'_j\big)} \\
&\times\frac{\prod_{j\le n}\prod_{i\le N-n}\Upsilon\big(-\frac{N}{N-n}\beta+\gamma_i-\delta_{n,j}\big)}{\prod_{i\le N-n}\prod_{j\le n'}\Upsilon\big(\frac{Q}{2}-\mu-\frac{n}{N-n}\beta-\beta'+\gamma_i-\delta_{n',j}\big)} \\
&\times\frac{\prod_{i\le N-n'}\prod_{j\le n'}\Upsilon\big(\frac{N}{N-n'}\beta'-\gamma'_i+\delta_{n',j}\big)}{\prod_{j\le N-n'}\prod_{i\le n}\Upsilon\big(\frac{Q}{2}-\mu+\beta+\frac{n'}{N-n'}\beta'-\gamma'_j+\delta_{n,i}\big)} \\
&\times\frac{\prod_{i<j\le n}\Upsilon(\delta_{n,i}-\delta_{n,j})\prod_{i<j\le n'}\Upsilon(\delta_{n',j}-\delta_{n',i})}{\prod_{i\le n}\prod_{j\le n'}\Upsilon\big(\frac{Q}{2}-\mu+\beta-\beta'+\delta_{n,i}-\delta_{n',j}\big)}
\end{aligned}
$$

- The last line has zero of order $\frac{1}{2}(n(n-1)+n'(n'-1))$

Specializing to the "hook" states we find

$$C_{FL}\big(2\vec{Q}-\alpha,(\tfrac{Q}{2}-\mu)N\omega_1,\alpha'\big)=\Big[\pi\bar{\mu}\gamma(b^2)b^{2-2b^2}\Big]^{\langle\alpha-\alpha',\rho\rangle/b}\big(\Upsilon(b)\big)^{N-1}\Upsilon\big(N(\tfrac{Q}{2}-\mu)\big)$$

$$\times\frac{\prod_{i<j\le N-n}\Upsilon\big(\gamma_i-\gamma_j\big)\prod_{i<j\le N-n'}\Upsilon\big(\gamma'_j-\gamma'_i\big)}{\prod_{i\le N-n}\prod_{j\le N-n'}\Upsilon\big(\frac{Q}{2}-\mu-\frac{n}{N-n}\beta+\frac{n'}{N-n'}\beta'+\gamma_i-\gamma'_j\big)}$$

$$\times\frac{\prod_{j\le n}\prod_{i\le N-n}\Upsilon\big(-\frac{N}{N-n}\beta+\gamma_i-\delta_{n,j}\big)}{\prod_{i\le N-n}\prod_{j\le n'}\Upsilon\big(\frac{Q}{2}-\mu-\frac{n}{N-n}\beta-\beta'+\gamma_i-\delta_{n',j}\big)}$$

$$\times\frac{\prod_{i\le N-n'}\prod_{j\le n'}\Upsilon\big(\frac{N}{N-n'}\beta'-\gamma'_i+\delta_{n',j}\big)}{\prod_{j\le N-n'}\prod_{i\le n}\Upsilon\big(\frac{Q}{2}-\mu+\beta+\frac{n'}{N-n'}\beta'-\gamma'_j+\delta_{n,i}\big)}$$

$$\times\frac{\prod_{i<j\le n}\Upsilon(\delta_{n,i}-\delta_{n,j})\prod_{i<j\le n'}\Upsilon(\delta_{n',j}-\delta_{n',i})}{\prod_{i\le n}\prod_{j\le n'}\Upsilon\big(\frac{Q}{2}-\mu+\beta-\beta'+\delta_{n,i}-\delta_{n',j}\big)}$$

- The last line has zero of order $\frac{1}{2}(n(n-1)+n'(n'-1))$
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- The last line has zero of order $\frac{1}{2}(n(n-1)+n'(n'-1))$
- Imposing $\boxed{\beta'=\beta-\mu}$ the order of the zero becomes $\frac{(n-n')^2-n-n'}{2}$
- Requiring a pole of order n', one obtain $\boxed{n'=n-1}$

Conclusion

- The fusion rules for semi-degenerate states show that states in the internal channels are the ones expected from the gauge theory analysis
- The space of states in any internal channel has the dimension of the Coulomb branch of the related gauge multiplet
- The residues of the FL formula provide the 3-point function for 2 semi-degenerate states and a simple puncture. This 3-point functions reproduces the 1-loop part of the gauge theory partition function. This provides the precise dictionary that relates gauge theories with tails and Toda CFT
- For the $SU(2) \times SU(3)$ quiver, we have shown that the W_3 conformal blocks reproduce the $SU(2)$ instantons, once we take into account the constraints that relate the primaries
- Some future directions:
  - Non-local operators in quiver tails
  - Generalization to non-conformal theories
  - Possible different description coupling Toda CFT's with different ranks