

Wilson Loops in $\mathcal{N} = 2$ Gauge Theories, Matrix Models and Holography

Filippo Passerini
Humboldt University Berlin

Problemi Attuali di Fisica Teorica
Vietri sul Mare
March 31, 2012

Based on: 1106.5763 F.P. and K. Zarembo

Introduction

- **Supersymmetry** provides a simplified framework to study Quantum Field Theory

Introduction

- **Supersymmetry** provides a simplified framework to study Quantum Field Theory
- **$\mathcal{N} = 4$ SYM**: many novel tools for the weak and **strong** coupling dynamics of quantum fields (AdS/CFT, Pestun Localization, Integrability, ...)

Introduction

- **Supersymmetry** provides a simplified framework to study Quantum Field Theory
- **$\mathcal{N} = 4$ SYM**: many novel tools for the weak and **strong** coupling dynamics of quantum fields (AdS/CFT, Pestun Localization, Integrability, ...)

Next Step: reduce the symmetry \Rightarrow increase complexity of the theory

Introduction

- **Supersymmetry** provides a simplified framework to study Quantum Field Theory
- **$\mathcal{N} = 4$ SYM**: many novel tools for the weak and **strong** coupling dynamics of quantum fields (AdS/CFT, Pestun Localization, Integrability, ...)

Next Step: reduce the symmetry \Rightarrow increase complexity of the theory

$\mathcal{N} = 2$ Gauge Theories

Interesting features: matter fields, non-trivial instanton dynamics, ...

Introduction

- **Supersymmetry** provides a simplified framework to study Quantum Field Theory
- $\mathcal{N} = 4$ **SYM**: many novel tools for the weak and **strong** coupling dynamics of quantum fields (AdS/CFT, Pestun Localization, Integrability, ...)

Next Step: reduce the symmetry \Rightarrow increase complexity of the theory

$\mathcal{N} = 2$ Gauge Theories

Interesting features: matter fields, non-trivial instanton dynamics, ...

Also for these theories, the strong coupling dynamics can be studied using

- **Pestun Localization**: reduce the field theory path integral to a matrix integral, for any value of the gauge coupling. VEV of certain non-local operators can be computed using a matrix model. **Matrix model is known for any $\mathcal{N} = 2$ theory.**
- **AdS/CFT**: relates the strong coupling of the gauge theory to string theory in a certain background. **The string dual is not known for most of the $\mathcal{N} = 2$ theory.**

Introduction

- **Supersymmetry** provides a simplified framework to study Quantum Field Theory
- $\mathcal{N} = 4$ **SYM**: many novel tools for the weak and **strong** coupling dynamics of quantum fields (AdS/CFT, Pestun Localization, Integrability, ...)

Next Step: reduce the symmetry \Rightarrow increase complexity of the theory

$\mathcal{N} = 2$ Gauge Theories

Interesting features: matter fields, non-trivial instanton dynamics, ...

Also for these theories, the strong coupling dynamics can be studied using

- **Pestun Localization**: reduce the field theory path integral to a matrix integral, for any value of the gauge coupling. VEV of certain non-local operators can be computed using a matrix model. **Matrix model is known for any $\mathcal{N} = 2$ theory.**
- **AdS/CFT**: relates the strong coupling of the gauge theory to string theory in a certain background. **The string dual is not known for most of the $\mathcal{N} = 2$ theory.**

Use the exact results of Pestun to study the string dual of $\mathcal{N} = 2$ theories !!

Outline

- introduction
- Wilson loop in $\mathcal{N} = 2$ gauge theory
- Pestun localization: VEV of a Wilson loop from matrix model
- Wilson loop in $\mathcal{N} = 4$ SYM
- Wilson loop in $\mathcal{N} = 2$ SCYM
 - weak coupling
 - strong coupling
- conclusion

$\mathcal{N} = 2$ Gauge Theory

[Salam, Strathdee][Fayet][Sohnius, Stelle, West][...]

can be constructed using the following building blocks

- $\mathcal{N} = 2$ vector multiplet $(A_\mu, \Phi_1, \Phi_2, \lambda_1, \lambda_2)$ in the adjoint rep. of the gauge group
- $\mathcal{N} = 2$ hypermultiplet $(\Phi_3, \Phi_4, \Phi_5, \Phi_6, \chi_3, \chi_4)$ in some rep. of the gauge group

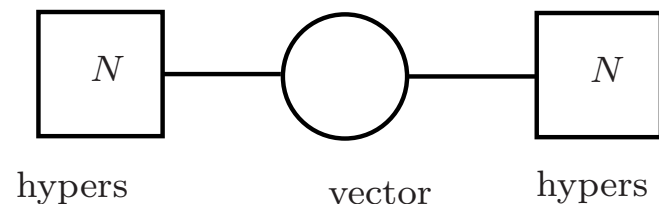
$\mathcal{N} = 2$ Gauge Theory

[Salam, Strathdee][Fayet][Sohnius, Stelle, West][...]

can be constructed using the following building blocks

- $\mathcal{N} = 2$ vector multiplet $(A_\mu, \Phi_1, \Phi_2, \lambda_1, \lambda_2)$ in the adjoint rep. of the gauge group
- $\mathcal{N} = 2$ hypermultiplet $(\Phi_3, \Phi_4, \Phi_5, \Phi_6, \chi_3, \chi_4)$ in some rep. of the gauge group

An interesting example is the $\mathcal{N} = 2$ $SU(N)$ SCYM:



- $2N$ hypermultiplets coupled to $SU(N)$ vector multiplet, i.e. $N_F = 2N_C$
 $\Rightarrow \beta$ -function vanishes at any loop, conformal QFT [Howe, Stelle, West]
- it is expected to have a string theory dual, that is not know yet. The string dual should have a $SO(2, 4)$ bosonic symmetry

SUSY Wilson Loops in $\mathcal{N} = 2$ Gauge Theory

in a theory with **at least** a vector multiplet $(A_\mu, \Phi_1, \Phi_2, \lambda_1, \lambda_2)$, one can define

$$W_R(C) = \frac{1}{N} \text{tr}_R \mathcal{P} \exp \left[\int_C ds (iA_\mu(x)\dot{x}^\mu + n_I \Phi_I(x)|\dot{x}|) \right]$$

- R a representation of the gauge group
- $\frac{1}{2}$ *BPS* observable when $C = \text{Circle}$

SUSY Wilson Loops in $\mathcal{N} = 2$ Gauge Theory

in a theory with **at least** a vector multiplet $(A_\mu, \Phi_1, \Phi_2, \lambda_1, \lambda_2)$, one can define

$$W_R(C) = \frac{1}{N} \text{tr}_R \mathcal{P} \exp \left[\int_C ds (iA_\mu(x)\dot{x}^\mu + n_I \Phi_I(x)|\dot{x}|) \right]$$

- R a representation of the gauge group
- $\frac{1}{2}$ **BPS observable** when $C = \text{Circle}$

The VEV SUSY Wilson loops in $\mathcal{N} = 2$ can be computed using a **matrix model**

[Pestun]

$$\langle W_R(\text{Circle}) \rangle = \text{Matrix Model}$$

Exact result: weak and strong coupling

Study the VEV of the Wilson loops at weak and strong coupling to investigate $\mathcal{N} = 2$ **gauge theories and their gravity duals !!**

Pestun Localization

- from quantum fields (infinite DOF) to matrices (finite DOF)

Pestun Localization

- from quantum fields (infinite DOF) to matrices (finite DOF)

Basic idea: Given a partition function for an $\mathcal{N} = 2$ gauge theory

$$Z = \int D\Psi e^{-S[\Psi]} \quad \Psi = \text{fields}$$

- choose a fermionic sym. Q , i.e. $QS[\Psi] = 0$
- choose a functional $V[\Psi]$, such that $Q^2V[\Psi] = 0$

Pestun Localization

- from quantum fields (infinite DOF) to matrices (finite DOF)

Basic idea: Given a partition function for an $\mathcal{N} = 2$ gauge theory

$$Z = \int D\Psi e^{-S[\Psi]} \quad \Psi = \text{fields}$$

- choose a fermionic sym. Q , i.e. $QS[\Psi] = 0$
- choose a functional $V[\Psi]$, such that $Q^2V[\Psi] = 0$

A deformation $QV[\Psi]$ of the action does not change the path integral

$$Z(t) = \int D\Psi e^{-(S[\Psi]+tQV[\Psi])} \quad \frac{dZ(t)}{dt} = 0$$

- $t = 0$ the original path integral (hard)
- $t = \infty$ saddle point techniques (easy)

Pestun Localization

- from quantum fields (infinite DOF) to matrices (finite DOF)

Basic idea: Given a partition function for an $\mathcal{N} = 2$ gauge theory

$$Z = \int D\Psi e^{-S[\Psi]} \quad \Psi = \text{fields}$$

- choose a fermionic sym. Q , i.e. $QS[\Psi] = 0$
- choose a functional $V[\Psi]$, such that $Q^2V[\Psi] = 0$

A deformation $QV[\Psi]$ of the action does not change the path integral

$$Z(t) = \int D\Psi e^{-(S[\Psi]+tQV[\Psi])} \quad \frac{dZ(t)}{dt} = 0$$

$-t = 0$ the original path integral (hard)

$-t = \infty$ saddle point techniques (easy)

since $Z(0) = Z(\infty)$, use saddle point techniques to exactly compute a path integral !

- gauge theory on S^4
- choose a charge Q that preserves the Wilson loop
- saddle points Ψ_0 satisfy $QV[\Psi_0] = 0$
 - S^4 w/o poles, the adjoint scalar assumes a constant value $\langle \Phi \rangle = M$, and all the other fields vanish
 - at the poles, there are point like instantons
- quantum contributions are encoded by quadratic fluctuations around the saddle points

- gauge theory on S^4
- choose a charge Q that preserves the Wilson loop
- saddle points Ψ_0 satisfy $QV[\Psi_0] = 0$
 - S^4 w/o poles, the adjoint scalar assumes a constant value $\langle \Phi \rangle = M$, and all the other fields vanish
 - at the poles, there are point like instantons
- quantum contributions are encoded by quadratic fluctuations around the saddle points

$Z(0) = Z(\infty)$, integration over fields is equivalent to integration over saddle points

$$Z = \int D\Psi e^{-S[\Psi]} = \int DM e^{-S[M]} Z_{1\text{-loop}}(M) Z_{\text{inst}}(M)$$

- $S[M] = -\frac{8\pi^2}{g^2} \text{Tr}(M^2)$
- $Z_{1\text{-loop}}(M)$ and $Z_{\text{inst}}(M)$ depend on the specific $\mathcal{N} = 2$ theory

- gauge theory on S^4
- choose a charge Q that preserves the Wilson loop
- saddle points Ψ_0 satisfy $QV[\Psi_0] = 0$
 - S^4 w/o poles, the adjoint scalar assumes a constant value $\langle \Phi \rangle = M$, and all the other fields vanish
 - at the poles, there are point like instantons
- quantum contributions are encoded by quadratic fluctuations around the saddle points

$Z(0) = Z(\infty)$, integration over fields is equivalent to integration over saddle points

$$Z = \int D\Psi e^{-S[\Psi]} = \int DM e^{-S[M]} Z_{1\text{-loop}}(M) Z_{\text{inst}}(M)$$

- $S[M] = -\frac{8\pi^2}{g^2} \text{Tr}(M^2)$
- $Z_{1\text{-loop}}(M)$ and $Z_{\text{inst}}(M)$ depend on the specific $\mathcal{N} = 2$ theory

$\frac{1}{2}$ BPS Wilson loop can be computed as an observable of the matrix model

$$\langle W_R(\text{Circle}) \rangle = \langle \frac{1}{N} \text{tr}_R e^{2\pi M} \rangle_{\text{Matrix Model}}$$

Simplest case: $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM is an $\mathcal{N} = 2$ vector multiplet coupled to an adjoint massless $\mathcal{N} = 2$ hyper

- for this theory $Z_{1\text{-loop}} = Z_{\text{inst}} = 1$
- therefore the associated matrix model is the Gaussian matrix model
[Erickson, Semenoff, Zarembo][Drukker, Gross]

$$Z_{\text{Gauss}} = \int DM e^{-\frac{8\pi^2}{g^2} \text{Tr}(M^2)}$$

Simplest case: $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM is an $\mathcal{N} = 2$ vector multiplet coupled to an adjoint massless $\mathcal{N} = 2$ hyper

- for this theory $Z_{1\text{-loop}} = Z_{\text{inst}} = 1$
- therefore the associated matrix model is the Gaussian matrix model
[Erickson, Semenoff, Zarembo][Drukker, Gross]

$$Z_{\text{Gauss}} = \int DM e^{-\frac{8\pi^2}{g^2} \text{Tr}(M^2)}$$

Diagonalizing M

$$Z_{\text{Gauss}} = \int d^{N-1}a \prod_{i < j} (a_i - a_j)^2 e^{-\frac{8\pi^2}{g^2} \sum_i a_i^2}$$

and the expectation value of the circular Wilson loop

$$\langle W(C_{\text{circle}}) \rangle_{\mathcal{N}=4} = \left\langle \frac{1}{N} \sum_i e^{2\pi a_i} \right\rangle_{\text{Gauss}}$$

Large N limit: $N \rightarrow \infty$ with $\lambda = Ng^2$ fixed

$$Z_{\text{Gauss}} = \int d^{N-1}a e^{-NS(a)} \quad S(a) = \sum_i \frac{8\pi^2}{\lambda} a_i^2 - \frac{1}{N} \sum_{i<j} \ln(a_i - a_j)^2$$

Saddle point equation: $\frac{8\pi^2}{\lambda} a_i - \frac{1}{N} \sum_{j \neq i} \left(\frac{1}{a_i - a_j} \right) = 0$

Large N limit: $N \rightarrow \infty$ with $\lambda = Ng^2$ fixed

$$Z_{\text{Gauss}} = \int d^{N-1}a e^{-NS(a)} \quad S(a) = \sum_i \frac{8\pi^2}{\lambda} a_i^2 - \frac{1}{N} \sum_{i<j} \ln(a_i - a_j)^2$$

Saddle point equation: $\frac{8\pi^2}{\lambda} a_i - \frac{1}{N} \sum_{j \neq i} \left(\frac{1}{a_i - a_j} \right) = 0$

- introducing the eigenvalue distribution $\rho(x) = \frac{1}{N} \sum_i \delta(x - a_i)$ defined in $(-\mu, \mu)$
- saddle point equation

$$\int_{-\mu}^{\mu} dy \rho(y) \left(\frac{1}{x - y} \right) = \frac{8\pi^2}{\lambda} x \quad \int_{-\mu}^{\mu} \rho(x) = 1$$

- Wilson loop VEV

$$\langle W(C_{\text{circle}}) \rangle_{\mathcal{N}=4} = \int_{-\mu}^{\mu} \rho(x) e^{2\pi x}$$

- the density is the Wigner semicircle $\rho(x) = \frac{8\pi}{\lambda} \sqrt{\mu^2 - x^2}$, with $\mu = \frac{\sqrt{\lambda}}{2\pi}$

Large N limit: $N \rightarrow \infty$ with $\lambda = Ng^2$ fixed

$$Z_{\text{Gauss}} = \int d^{N-1}a e^{-NS(a)} \quad S(a) = \sum_i \frac{8\pi^2}{\lambda} a_i^2 - \frac{1}{N} \sum_{i<j} \ln(a_i - a_j)^2$$

Saddle point equation: $\frac{8\pi^2}{\lambda} a_i - \frac{1}{N} \sum_{j \neq i} \left(\frac{1}{a_i - a_j} \right) = 0$

- introducing the eigenvalue distribution $\rho(x) = \frac{1}{N} \sum_i \delta(x - a_i)$ defined in $(-\mu, \mu)$
- saddle point equation

$$\int_{-\mu}^{\mu} dy \rho(y) \left(\frac{1}{x - y} \right) = \frac{8\pi^2}{\lambda} x \quad \int_{-\mu}^{\mu} \rho(x) = 1$$

- Wilson loop VEV

$$\langle W(C_{\text{circle}}) \rangle_{\mathcal{N}=4} = \int_{-\mu}^{\mu} \rho(x) e^{2\pi x}$$

- the density is the Wigner semicircle $\rho(x) = \frac{8\pi}{\lambda} \sqrt{\mu^2 - x^2}$, with $\mu = \frac{\sqrt{\lambda}}{2\pi}$

Therefore the VEV of the circular Wilson loop in the 't Hooft limit is

$$\langle W(C_{\text{circle}}) \rangle_{\mathcal{N}=4} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

- Weak coupling $\lambda \ll 1$

$$\langle W(C_{\text{circle}}) \rangle_{\mathcal{N}=4} = \sum_{n=0}^{\infty} \frac{(\lambda/4)^n}{n!(n+1)!} = 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \frac{\lambda^3}{9216} + \dots$$

in agreement with **perturbation theory**

- Strong coupling $\lambda \gg 1$

$$\langle W(C_{\text{circle}}) \rangle_{\mathcal{N}=4} \simeq \sqrt{\frac{2}{\pi}} \lambda^{-3/4} e^{\sqrt{\lambda}}$$

in agreement with the result obtained in the **string theory dual**
(i.e. IIB strings $AdS_5 \times S^5$)

Wilson loops in AdS/CFT

- consider a **conformal theory in 4D** $\Rightarrow SO(4, 2)$ is part of the bosonic symmetry
- the dual background should include an **AdS_5 factor**

The gauge theory lives on the boundary of the AdS_5 and the Wilson loop is associated to an open string that ends on the loop [Maldacena][Rey, Yee]

$$\langle W(C) \rangle = \int_{\partial X=C} DX e^{-T S[X]}$$

- T is the **string tension**, $S[X]$ is the **area functional**

Wilson loops in AdS/CFT

- consider a **conformal theory in 4D** $\Rightarrow SO(4, 2)$ is part of the bosonic symmetry
- the dual background should include an **AdS_5 factor**

The gauge theory lives on the boundary of the AdS_5 and the Wilson loop is associated to an open string that ends on the loop [Maldacena][Rey, Yee]

$$\langle W(C) \rangle = \int_{\partial X=C} DX e^{-T S[X]}$$

- T is the **string tension**, $S[X]$ is the **area functional**

for the case $C = \text{circle}$, in the regime $T \rightarrow \infty$ [Drukker, Gross]

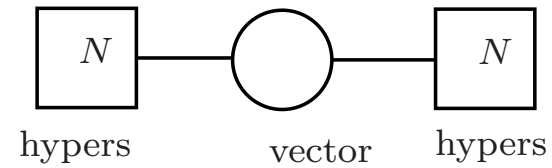
$$\langle W(C) \rangle_{\mathcal{N}=4} \simeq K T^{-3/2} e^{2\pi T}$$

$\mathcal{N}=4$ SYM: the dual string theory is IIB on $AdS_5 \times S^5$ with tension

$$T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$$

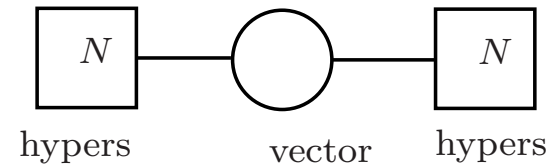
and the string theory computation is in **agreement with the matrix model result !!**

$\mathcal{N} = 2$ $SU(N)$ SCYM:



We can probe the unknown string dual computing the VEV of the circular Wilson loop in the regime $N \rightarrow \infty$ and $\lambda \gg 1$.

$\mathcal{N} = 2$ $SU(N)$ SCYM:



We can probe the unknown string dual computing the VEV of the circular Wilson loop in the regime $N \rightarrow \infty$ and $\lambda \gg 1$.

Partition function for $\mathcal{N} = 2$ $SU(N)$ SCYM:

[Pestun]

$$Z = \int d^{N-1}a \prod_{i < j} (a_i - a_j)^2 e^{-\frac{8\pi^2}{g^2} \sum_i a_i^2} \mathcal{Z}_{1\text{-loop}}(a) |\mathcal{Z}_{\text{inst}}(a; g^2)|^2$$

- 1-loop contribution

$$\mathcal{Z}_{1\text{-loop}} = \frac{\prod_{i < j} H^2(a_i - a_j)}{\prod_i H^{2N}(a_i)}$$

where $H(x) = e^{-(1+\gamma)x^2} G(1+ix)G(1-ix)$ and $G(z)$ Barnes function

- instanton contribution

$$\mathcal{Z}_{\text{inst}}(a; g^2) = 1 + w_1(a) e^{-\frac{8\pi^2}{\lambda} N} + w_2(a) e^{-\left(\frac{8\pi^2}{\lambda} N\right)^2} + \dots \quad N \rightarrow \infty, w_1 \propto \sqrt{N}$$

Large N limit:

$$\mathcal{Z}_{\text{inst}}(a; g^2) = 1$$

no instanton contribution

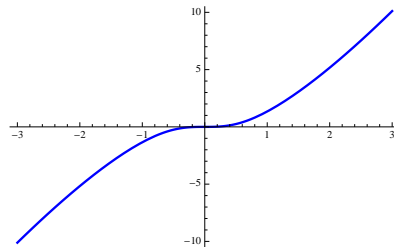
- effective action

$$S(a) = \sum_i \left(\frac{8\pi^2}{\lambda} a_i^2 + 2 \ln H(a_i) \right) - \frac{1}{N} \sum_{i < j} \left(\ln (a_i - a_j)^2 + 2 \ln H(a_i - a_j) \right)$$

- saddle point equation

$$\frac{8\pi^2}{\lambda} a_i - K(a_i) - \frac{1}{N} \sum_{j \neq i} \left(\frac{1}{a_i - a_j} - K(a_i - a_j) \right) = 0$$

where $K(x) = -\frac{H'(x)}{H(x)}$



$$K(x) \approx 2x \ln x \quad (x \rightarrow +\infty)$$

$$K(x) \approx 2\zeta(3)x^3 \quad (x \rightarrow 0)$$

- saddle point equation in the continuum limit

$$\int_{-\mu}^{\mu} dy \rho(y) \left(\frac{1}{x-y} - K(x-y) \right) = \frac{8\pi^2}{\lambda} x - K(x)$$

$$\rho(x) = ?$$

- not easy to solve this integral equation exactly. Focus on the **weak and strong coupling regime**.

- saddle point equation in the continuum limit

$$\int_{-\mu}^{\mu} dy \rho(y) \left(\frac{1}{x-y} - K(x-y) \right) = \frac{8\pi^2}{\lambda} x - K(x)$$

$$\rho(x) = ?$$

- not easy to solve this integral equation exactly. Focus on the **weak and strong coupling regime**.

Weak coupling regime $\lambda \ll 1$

- strong attractive potential for the eigenvalues
 \Rightarrow the eigenvalues are in the interval $(-\mu, \mu)$ with $\mu \ll 1$
- therefore $K(x) = -2 \sum_{n=1}^{\infty} (-1)^n \zeta(2n+1) x^{2n+1}$ (Taylor expansion)
- **truncating the expansion** of $K(x)$ is possible to obtain an **approximate expression** for $\rho(x)$ and for the VEV of the Wilson loop

$$\rho(x) = \frac{8\pi}{\lambda} \sqrt{\mu^2 - x^2} - \frac{1}{\pi^2} \int_{-\mu}^{\mu} \frac{dy}{x-y} \sqrt{\frac{\mu^2 - x^2}{\mu^2 - y^2}} \int dz \rho(z) (K(y-z) - K(y))$$

Perturbative scheme:

- lowest order $K(x) = 2\zeta(3)x^3$

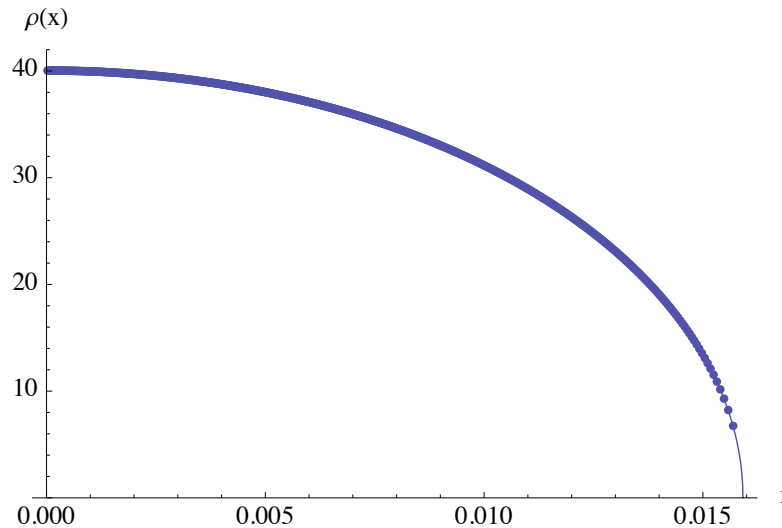
- Use the truncated $K(x)$ to compute $\rho(x)$

$$\rho_{m_2, \mu, \lambda}(x) = \left(\frac{8\pi}{\lambda} + \frac{6\zeta(3)m_2}{\pi} \right) \sqrt{\mu^2 - x^2} \text{ where } m_2 = \int_{-\mu}^{\mu} dz \rho(z) z^2$$

- Consistency condition $m_2 = \int_{-\mu}^{\mu} dz \rho_{m_2, \mu, \lambda}(z) z^2 \Rightarrow m_2 = m_2(\mu, \lambda)$

- normalization $\int_{-\mu}^{\mu} \rho_{\mu, \lambda}(z) dz = 1 \Rightarrow \mu = \frac{\sqrt{\lambda}}{2\pi} - \frac{3\zeta(3)\lambda^{5/2}}{256\pi^5} + \dots$

- therefore we obtain the approximated density depending on λ , $\rho_\lambda(x)$



- the Wilson loop VEV

$$\langle W(C_{\text{circle}}) \rangle = 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \left(\frac{1}{9216} - \frac{3\zeta(3)}{512\pi^4} \right) \lambda^3 + \dots$$

first difference respect $\mathcal{N} = 4$ is at $\mathcal{O}(\lambda^3)$, in agreement with perturbative calculation

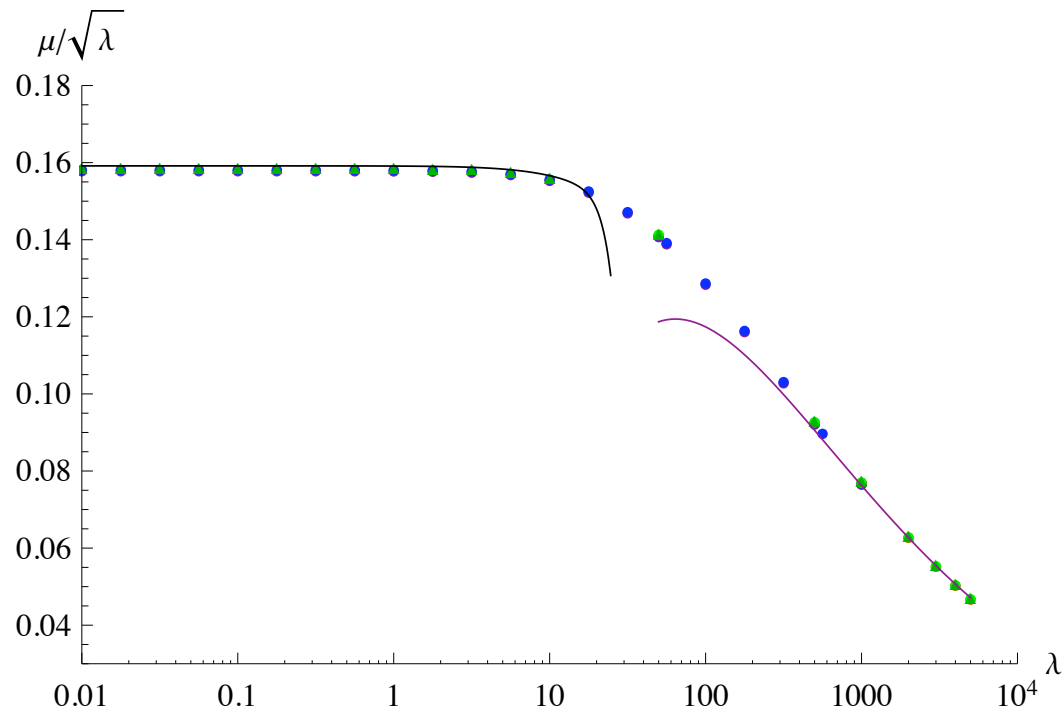
[Andree, Young]

This perturbative scheme can be pushed to **arbitrary high order in λ**

$$\begin{aligned}
\langle W(C_{\text{circle}}) \rangle &= 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \left(\frac{1}{9216} - \frac{3\zeta(3)}{512\pi^4} \right) \lambda^3 \\
&+ \left(\frac{1}{737280} - \frac{2\pi^2\zeta(3) - 15\zeta(5)}{4096\pi^6} \right) \lambda^4 \\
&+ \left(\frac{1}{88473600} - \frac{3\pi^4\zeta(3) - 65\pi^2\zeta(5) - 12(9\zeta(3)^2 - 35\zeta(7))}{196608\pi^8} \right) \lambda^5 \\
&+ \left(\frac{1}{14863564800} + \frac{-2\pi^2\zeta(3) + 85\zeta(5)}{7864320\pi^6} \right. \\
&\quad \left. + \frac{\pi^2(180\zeta(3)^2 - 637\zeta(7)) - 45(60\zeta(3)\zeta(5) - 91\zeta(9))}{3145728\pi^{10}} \right) \lambda^6 \\
&+ \left(\frac{1}{3329438515200} + \frac{-\pi^2\zeta(3) + 70\zeta(5)}{377487360\pi^6} \right. \\
&\quad + \frac{3\pi^2(108\zeta(3)^2 - 343\zeta(7)) - 126(110\zeta(3)\zeta(5) - 153\zeta(9))}{150994944\pi^{10}} \\
&\quad \left. - \frac{27(360\zeta(3)^3 - 1900\zeta(5)^2 - 3360\zeta(3)\zeta(7) + 4697\zeta(11))}{150994944\pi^{12}} \right) \lambda^7 \\
&+ O(\lambda^8)
\end{aligned}$$

Strong coupling regime $\lambda \gg 1$

The effect of $k(x)$ change things significantly: strong repulsive central force and attractive 2-body interaction.



Limiting case, $\lambda = \infty$

- distribution $\rho_\infty(x)$ for eigenvalues $x \in (-\infty, \infty)$
- saddle point

$$\int_{-\infty}^{+\infty} dy \rho_\infty(y) \left(\frac{1}{x-y} - K(x-y) \right) = -K(x).$$

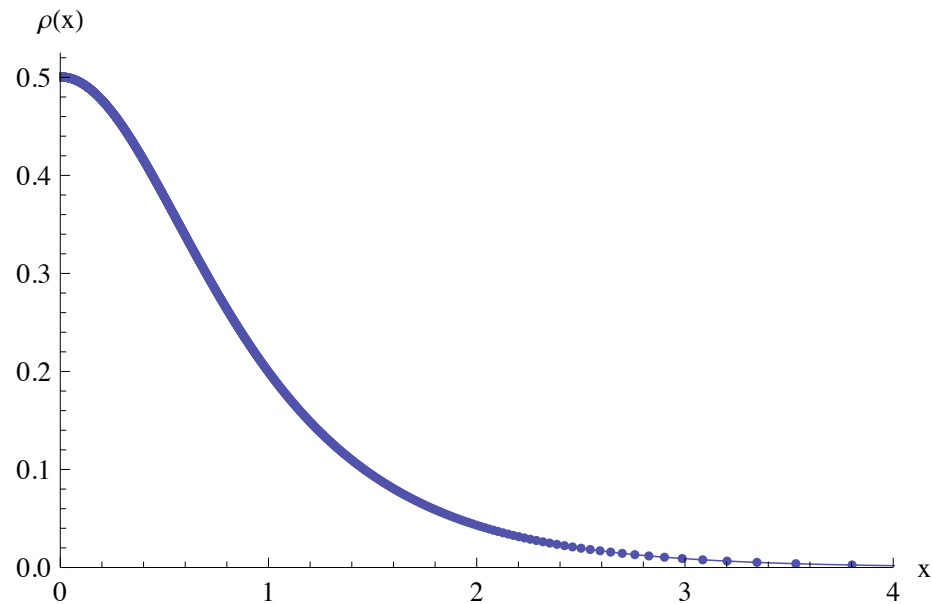
Limiting case, $\lambda = \infty$

- distribution $\rho_\infty(x)$ for eigenvalues $x \in (-\infty, \infty)$
- saddle point

$$\int_{-\infty}^{+\infty} dy \rho_\infty(y) \left(\frac{1}{x-y} - K(x-y) \right) = -K(x).$$

- Fourier transform: $\rho_\infty(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega x} \rho_\infty(\omega), \quad K(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega x} K(\omega)$

$$\rho_\infty(\omega) = \frac{1}{\cosh \omega} \quad \rho_\infty(x) = \frac{1}{2 \cosh \frac{\pi x}{2}}$$



$\lambda \gg 1, \lambda < +\infty$

- given an observable $\mathcal{O}(x)$, it results $\langle \mathcal{O}(x) \rangle_{\lambda \gg 1} \simeq \int_{-\infty}^{+\infty} dx \rho_{\infty}(x) \mathcal{O}(x)$ only if $\mathcal{O}(x) < e^{\frac{\pi x}{2}}$ at large x
- $\langle W(C) \rangle_{\lambda \gg 1} = \int_{-\mu}^{+\mu} dx \rho(x) e^{2\pi x} \simeq e^{2\pi\mu}$, therefore need to compute $\mu = \mu(\lambda)$

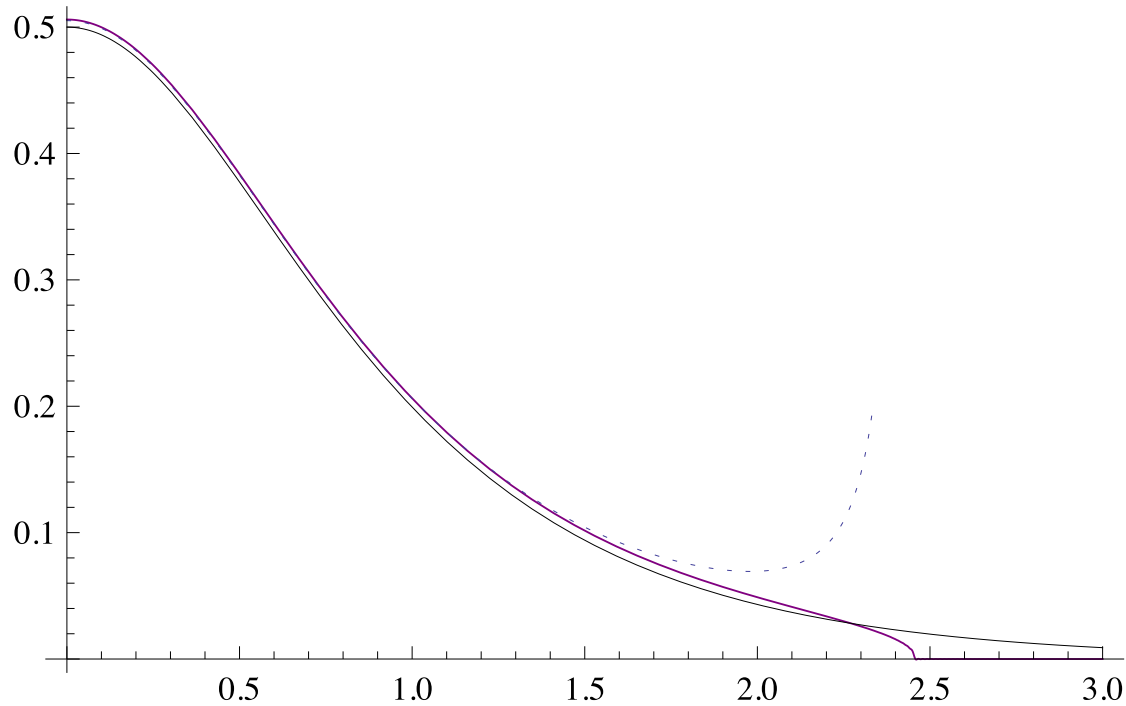
$\lambda \gg 1, \lambda < +\infty$

- given an observable $\mathcal{O}(x)$, it results $\langle \mathcal{O}(x) \rangle_{\lambda \gg 1} \simeq \int_{-\infty}^{+\infty} dx \rho_{\infty}(x) \mathcal{O}(x)$ only if $\mathcal{O}(x) < e^{\frac{\pi x}{2}}$ at large x
- $\langle W(C) \rangle_{\lambda \gg 1} = \int_{-\mu}^{+\mu} dx \rho(x) e^{2\pi x} \simeq e^{2\pi\mu}$, therefore need to compute $\mu = \mu(\lambda)$

Wiener-Hopf method: generalization of the Fourier transform for the case when an integral equation is defined on a semi-infinite interval

- in the regime $\lambda \gg 1$, the Fourier transform of the distribution $\rho(\omega)$

$$\rho(\omega) = \frac{1}{\cosh \omega} + \frac{2 \sinh^2 \frac{\omega}{2}}{\cosh \omega} F(\omega) + G_-(\omega) e^{i\mu\omega} \sum_{n=0}^{\infty} \frac{r_n e^{-\mu\nu_n}}{\omega + i\nu_n} (1 - F(-i\nu_n))$$

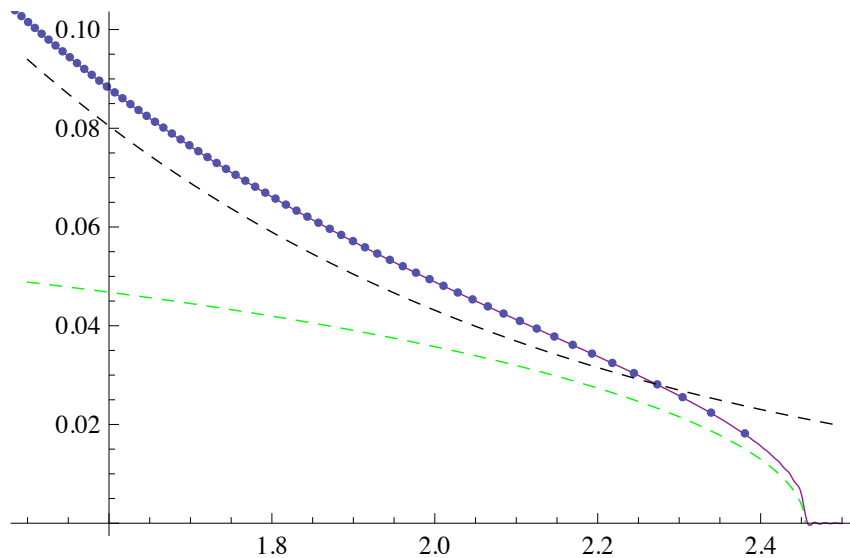


- Wilson loop VEV

$$\langle W(C_{\text{circle}}) \rangle_{\lambda \gg 1} = \rho(-2\pi i) = R \frac{\sqrt{\mu}}{\lambda} e^{2\pi\mu}$$

- normalization condition for $\rho(\omega)$ gives

$$C \sqrt{\mu} e^{\pi\mu/2} = \lambda$$



Qualitative estimates: near the endpoint μ the distribution is determined by the attractive force and the repulsive 2-body interaction

$$\Rightarrow \text{Wigner semicircle } \rho(x) \sim \frac{8\pi}{\lambda} \sqrt{\mu^2 - x^2}$$

- Wilson loop VEV $\langle W(C_{\text{circle}}) \rangle_{\lambda \gg 1} \simeq \frac{8\pi}{\lambda} \int^{\mu} dx \sqrt{\mu^2 - x^2} e^{2\pi x} = 2 \frac{\sqrt{\mu}}{\lambda} e^{2\pi\mu}$
- normalization condition

$$\int_{\mu}^{\infty} dx \rho_{\infty}(x) \simeq \int_{\mu-z_0}^{\mu} dx \frac{8\pi}{\lambda} \sqrt{\mu^2 - x^2} \quad \rightarrow \quad C\sqrt{\mu} e^{\pi\mu/2} = \lambda$$

combining the two expressions

$$\langle W(C_{\text{circle}}) \rangle_{\lambda \gg 1} = \text{const} \frac{\lambda^3}{(\ln \lambda)^{3/2}}$$

combining the two expressions

$$\langle W(C_{\text{circle}}) \rangle_{\lambda \gg 1} = \text{const} \frac{\lambda^3}{(\ln \lambda)^{3/2}}$$

that is equivalent to the string theory prediction

$$\langle W(C_{\text{circle}}) \rangle_{\lambda \gg 1} = K T^{-3/2} e^{2\pi T}$$

considering

$$T = \frac{3}{2\pi} \ln \lambda$$

Conclusions

- we compute the weak coupling VEV of the Wilson loop in $\mathcal{N} = 2$ SCYM at arbitrary high number of loops
- at strong coupling, the VEV of the loop has a stringy behaviour, although the tension of the string is related to the gauge theory coupling in an unusual way
- the strong coupling result carries information about the unknown string dual
- other interesting probes for the string dual are the Wilson loop in higher rank representation [Fraser, Kumar]
- there are Pestun matrix models for a large class of $\mathcal{N} = 2$ theories. Interesting to study other examples.