Wilson Loops in $\mathcal{N} = 2$ Gauge Theories, Matrix Models and Holography

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Also for these theories, the strong coupling dynamics can be studied using

- Pestun Localization: reduce the field theory path integral to a matrix integral, for any value of the gauge coupling. VEV of certain non-local operators can be computed using a matrix model. Matrix model is known for any $\mathcal{N}=2$ theory.
- AdS/CFT: relates the strong coupling of the gauge theory to string theory in a certain backgroud. The string dual is not known for most of the $\mathcal{N}=2$ theory.

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Use the exact results of Pestun to study the string dual of $\mathcal{N} = 2$ theories !!

Outline

- introduction
- Wilson loop in $\mathcal{N} = 2$ gauge theory
- Pestun localization: VEV of a Wilson loop from matrix model
- Wilson loop in $\mathcal{N} = 4$ SYM
- Wilson loop in $\mathcal{N} = 2$ SCYM
  - weak coupling
  - strong coupling
- conclusion

$\mathcal{N} = 2$ Gauge Theory

[Salam, Strathdee][Fayet][Sohnius, Stelle, West][...]

can be constructed using the following building blocks

- $\mathcal{N} = 2$ vector multiplet $(A_\mu, \Phi_1, \Phi_2, \lambda_1, \lambda_2)$ in the adjoint rep. of the gauge group
- $\mathcal{N} = 2$ hypermultiplet $(\Phi_3, \Phi_4, \Phi_5, \Phi_6, \chi_3, \chi_4)$ in some rep. of the gauge group

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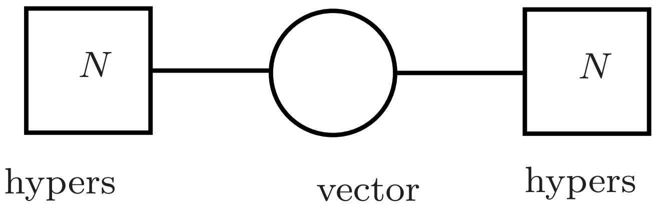
An interesting example is the $\mathcal{N} = 2$ $SU(N)$ SCYM:



- $2N$ hypermultiplets coupled to $SU(N)$ vector multiplet, i.e. $N_F = 2N_C$
  $\Rightarrow \beta$-function vanishes at any loop, conformal QFT [Howe, Stelle, West]
- it is expected to have a string theory dual, that is not know yet. The string dual should have a $SO(2,4)$ bosonic symmmetry

SUSY Wilson Loops in $\mathcal{N} = 2$ Gauge Theory

in a theory with at least a vector multiplet $(A_\mu, \Phi_1, \Phi_2, \lambda_1, \lambda_2)$, one can define

$$W_R(C) = \frac{1}{N}\,\mathrm{tr}\,_R \mathcal{P} \exp\left[\int_C ds\ \left(iA_\mu(x)\acute{x}^\mu + n_I \Phi_I(x)|\acute{x}|\right)\right]$$

- R a representation of the gauge group
- $\frac{1}{2}BPS$ observable when $C =$ Circle

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The VEV SUSY Wilson loops in $\mathcal{N}=2$ can be computed using a matrix model

[Pestun]

$$\langle W_R(\text{Circle})\rangle = \text{Matrix Model}$$

Exact result: weak and strong coupling

Study the VEV of the Wilson loops at weak and strong coupling to investigate $\mathcal{N}=2$ gauge theories and their gravity duals !!

Pestun Localization

- from quantum fields (infinite DOF) to matrices (finite DOF)

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Basic idea: Given a partition function for an $\mathcal{N} = 2$ gauge theory

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- choose a fermionic sym. Q, i.e. $QS[\Psi] = 0$
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A deformation $QV[\Psi]$ of the action does not change the path integral

$$Z(t) = \int D\Psi e^{-(S[\Psi]+tQV[\Psi])} \qquad\qquad \frac{dZ(t)}{dt} = 0$$

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since $Z(0) = Z(\infty)$, use saddle point techniques to exactly compute a path integral !

- gauge theory on S^4
- choose a charge Q that preserves the Wilson loop
- saddle points Ψ_0 satisfy $QV[\Psi_0] = 0$
  - S^4 w/o poles, the adjoint scalar assumes a constant value $\langle \Phi \rangle = M$, and all the other fields vanish
  - at the poles, there are point like instantons
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$Z(0) = Z(\infty)$, integration over fields is equivalent to integration over saddle points

$$Z = \int D\Psi e^{-S[\Psi]} = \int DM e^{-S[M]} Z_{\text{1-loop}}(M) Z_{\text{inst}}(M)$$

- $S[M] = -\frac{8\pi^2}{g^2} \mathrm{Tr}(M^2)$
- $Z_{\text{1-loop}}(M)$ and $Z_{\text{inst}}(M)$ depend on the specific $\mathcal{N} = 2$ theory

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$\frac{1}{2}$ BPS Wilson loop can be computed as an observable of the matrix model

$$\langle W_R(\text{Circle})\rangle = \langle \frac{1}{N}\text{tr}_R e^{2\pi M}\rangle_{\text{Matrix Model}}$$

Simplest case: $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM is an $\mathcal{N} = 2$ vector multiplet coupled to an adjoint massless $\mathcal{N} = 2$ hyper

- for this theory $Z_{\text{1-loop}} = Z_{\text{inst}} = 1$
- therefore the associated matrix model is the Gaussian matrix model [Erickson, Semenoff, Zarembo][Drukker, Gross]

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Diagonalizing M

$$Z_{\text{Gauss}} = \int d^{N-1}a \prod_{i<j}(a_i - a_j)^2 \, \mathrm{e}^{-\frac{8\pi^2}{g^2}\sum_i a_i^2}$$

and the expectation value of the circular Wilson loop

$$\langle W(C_{\text{circle}})\rangle_{\mathcal{N}=4} = \left\langle \frac{1}{N}\sum_i \mathrm{e}^{\,2\pi a_i} \right\rangle_{\text{Gauss}}$$

Large N limit: $N \to \infty$ with $\lambda = Ng^2$ fixed

$$Z_{\text{Gauss}} = \int d^{N-1}a \, e^{-NS(a)} \qquad S(a) = \sum_i \frac{8\pi^2}{\lambda} a_i^2 - \frac{1}{N} \sum_{i<j} \ln{(a_i - a_j)}^2$$

Saddle point equation: $\frac{8\pi^2}{\lambda} a_i - \frac{1}{N} \sum_{j \neq i} \left(\frac{1}{a_i - a_j} \right) = 0$

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- introducing the eigenvalue distribution $\rho(x) = \frac{1}{N} \sum_i \delta(x - a_i)$ defined in $(-\mu, \mu)$

  - saddle point equation

$$⨍_{-\mu}^{\mu} dy\, \rho(y) \left(\frac{1}{x-y} \right) = \frac{8\pi^2}{\lambda} x \qquad \int_{-\mu}^{\mu} \rho(x) = 1$$

  - Wilson loop VEV

$$\langle W(C_{\text{circle}}) \rangle_{\mathcal{N}=4} = \int_{-\mu}^{\mu} \rho(x) e^{2\pi x}$$

- the density is the Wigner semicircle $\rho(x) = \frac{8\pi}{\lambda} \sqrt{\mu^2 - x^2}$, with $\mu = \frac{\sqrt{\lambda}}{2\pi}$

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Therefore the VEV of the circular Wilson loop in the 't Hooft limit is

$$\langle W(C_{\text{circle}}) \rangle_{\mathcal{N}=4} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

- Weak coupling $\lambda \ll 1$

$$\langle W(C_{\text{circle}})\rangle_{\mathcal{N}=4} = \sum_{n=0}^{\infty} \frac{(\lambda/4)^n}{n!(n+1)!} = 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \frac{\lambda^3}{9216} + \dots$$

in agreement with perturbation theory

- Strong coupling $\lambda \gg 1$

$$\langle W(C_{\text{circle}})\rangle_{\mathcal{N}=4} \simeq \sqrt{\frac{2}{\pi}} \lambda^{-3/4} e^{\sqrt{\lambda}}$$

in agreement with the result obtained in the string theory dual (i.e. IIB strings $AdS_5 \times S^5$)

Wilson loops in AdS/CFT

- consider a conformal theory in $4D \Rightarrow SO(4,2)$ is part of the bosonic symmetry
- the dual background should include an AdS_5 factor

The gauge theory lives on the boundary of the AdS_5 and the Wilson loop is associated to an open string that ends on the loop [Maldacena][Rey, Yee]

$$\langle W(C) \rangle = \int_{\partial X = C} DX e^{-T\, S[X]}$$

- T is the string tension, $S[X]$ is the area functional

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for the case $C =$ circle, in the regime $T \to \infty$ [Drukker,Gross]

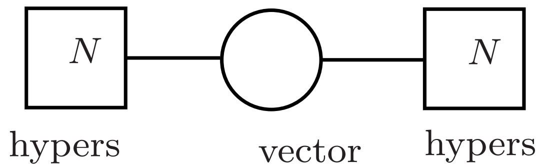
$$\langle W(C)\rangle_{\mathcal{N}=4} \simeq K T^{-3/2} e^{2\pi\, T}$$

$\mathcal{N}$=4 SYM: the dual string theory is IIB on $AdS_5 \times S^5$ with tension

$$T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$$

and the string theory computation is in agreement with the matrix model result !!

$\mathcal{N} = 2$ $SU(N)$ SCYM:



We can probe the unknown string dual computing the VEV of the circular Wilson loop in the regime $N \to \infty$ and $\lambda \gg 1$.

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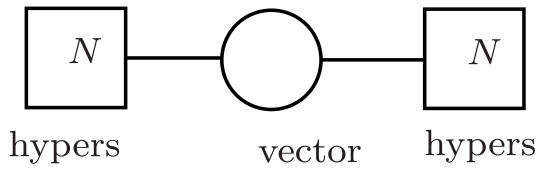

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Partition function for $\mathcal{N}=2$ $SU(N)$ SCYM: [Pestun]

$$Z = \int d^{N-1}a \prod_{i<j} (a_i - a_j)^2 \, \mathrm{e}^{-\frac{8\pi^2}{g^2} \sum_i a_i^2} \mathcal{Z}_{\text{1-loop}}(a) \left| \mathcal{Z}_{\text{inst}}(a; g^2) \right|^2$$

- 1-loop contribution

$$\mathcal{Z}_{\text{1-loop}} = \frac{\prod_{i<j} H^2(a_i - a_j)}{\prod_i H^{2N}(a_i)}$$

where $H(x) = \mathrm{e}^{-(1+\gamma)x^2} G(1+ix)G(1-ix)$ and $G(z)$ Barnes function

- instanton contribution

$$\mathcal{Z}_{\text{inst}}(a; g^2) = 1 + w_1(a) e^{-\frac{8\pi^2}{\lambda}N} + w_2(a) e^{-\left(\frac{8\pi^2}{\lambda}N\right)^2} + \ldots \qquad N \to \infty,\ w_1 \propto \sqrt{N}$$

Large N limit: $\mathcal{Z}_{\text{inst}}(a; g^2) = 1$ no instanton contribution

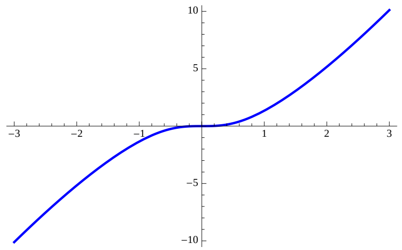
- effective action

$$S(a) = \sum_i \left(\frac{8\pi^2}{\lambda} a_i^2 + 2 \ln H(a_i) \right) - \frac{1}{N} \sum_{i<j} \left(\ln (a_i - a_j)^2 + 2 \ln H(a_i - a_j) \right)$$

- saddle point equation

$$\frac{8\pi^2}{\lambda} a_i - K(a_i) - \frac{1}{N} \sum_{j \neq i} \left(\frac{1}{a_i - a_j} - K(a_i - a_j) \right) = 0$$

where $K(x) = -\frac{H'(x)}{H(x)}$



$K(x) \approx 2x \ln x \qquad (x \to +\infty)$

$K(x) \approx 2\zeta(3) x^3 \qquad (x \to 0))$

- saddle point equation in the continuum limit

$$\fint_{-\mu}^{\mu} dy\, \rho(y) \left(\frac{1}{x-y} - K(x-y) \right) = \frac{8\pi^2}{\lambda} x - K(x)$$

$$\rho(x) = ?$$

- not easy to solve this integral equation exactly. Focus on the weak and strong coupling regime.

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- not easy to solve this integral equation exactly. Focus on the weak and strong coupling regime.

Weak coupling regime $\lambda \ll 1$

- strong attractive potential for the eigenvalues

  $\Rightarrow$ the eigenvalues are in the interval $(-\mu, \mu)$ with $\mu \ll 1$

- therefore $K(x) = -2\sum_{n=1}^{\infty}(-1)^n \zeta(2n+1) x^{2n+1}$ (Taylor expansion)

- truncating the expansion of $K(x)$ is possible to obtain an approximate expression for $\rho(x)$ and for the VEV of the Wilson loop

$$\rho(x) = \frac{8\pi}{\lambda}\sqrt{\mu^2 - x^2} - \frac{1}{\pi^2} \fint_{-\mu}^{\mu} \frac{dy}{x-y} \sqrt{\frac{\mu^2 - x^2}{\mu^2 - y^2}} \int dz\, \rho(z) \left(K(y-z) - K(y)\right)$$

Perturbative scheme:

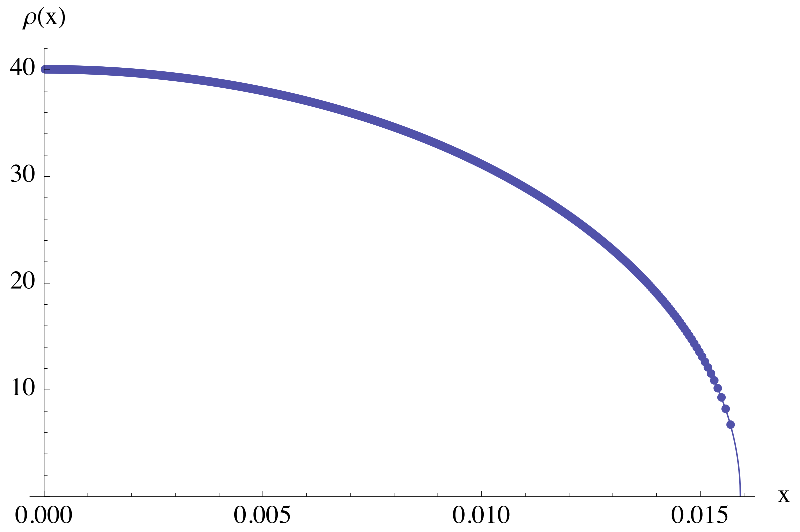
- lowest order $K(x) = 2\zeta(3)x^3$

- Use the truncated $K(x)$ to compute $\rho(x)$
  $\rho_{m_2,\mu,\lambda}(x) = \left(\frac{8\pi}{\lambda} + \frac{6\zeta(3)m_2}{\pi}\right)\sqrt{\mu^2 - x^2}$ where $m_2 = \int_{-\mu}^{\mu} dz\, \rho(z) z^2$

- Consistency condition $\quad m_2 = \int_{-\mu}^{\mu} dz\, \rho_{m_2,\mu,\lambda}(z) z^2 \quad \Rightarrow \quad m_2 = m_2(\mu, \lambda)$

- normalization $\quad \int_{\mu}^{-\mu} \rho_{\mu,\lambda}(z) = 1 \quad \Rightarrow \quad \mu = \frac{\sqrt{\lambda}}{2\pi} - \frac{3\zeta(3)\lambda^{5/2}}{256\pi^5} + \ldots$

- therefore we obtain the approximated density depending on λ, $\rho_\lambda(x)$



- the Wilson loop VEV

$$\langle W(C_{\text{circle}})\rangle = 1 + \frac{\lambda}{8} + \frac{\lambda^2}{192} + \left(\frac{1}{9216} - \frac{3\zeta(3)}{512\pi^4}\right)\lambda^3 + \ldots$$

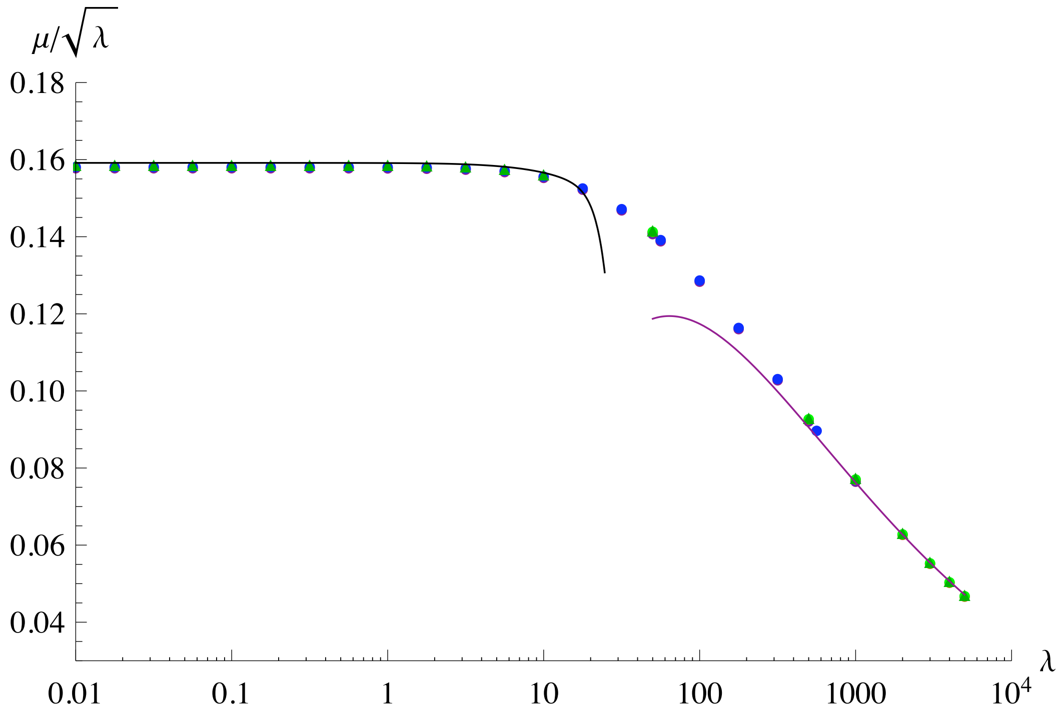
first difference respect $\mathcal{N} = 4$ is at $\mathcal{O}(\lambda^3)$, in agreement with perturbative calculation [Andree, Young]

This perturbative scheme can be pushed to arbitrary high order in λ

$$
\begin{aligned}
\langle W(C_{\rm circle})\rangle &= 1+\frac{\lambda}{8}+\frac{\lambda^2}{192}+\left(\frac{1}{9216}-\frac{3\zeta(3)}{512\pi^4}\right)\lambda^3 \\
&+ \left(\frac{1}{737280}-\frac{2\pi^2\zeta(3)-15\zeta(5)}{4096\pi^6}\right)\lambda^4 \\
&+ \left(\frac{1}{88473600}-\frac{3\pi^4\zeta(3)-65\pi^2\zeta(5)-12\left(9\zeta(3)^2-35\zeta(7)\right)}{196608\pi^8}\right)\lambda^5 \\
&+ \left(\frac{1}{14863564800}+\frac{-2\pi^2\zeta(3)+85\zeta(5)}{7864320\pi^6}\right. \\
&\quad \left.+\frac{\pi^2\left(180\zeta(3)^2-637\zeta(7)\right)-45(60\zeta(3)\zeta(5)-91\zeta(9))}{3145728\pi^{10}}\right)\lambda^6 \\
&+ \left(\frac{1}{3329438515200}+\frac{-\pi^2\zeta(3)+70\zeta(5)}{377487360\pi^6}\right. \\
&\quad +\frac{3\pi^2\left(108\zeta(3)^2-343\zeta(7)\right)-126(110\zeta(3)\zeta(5)-153\zeta(9))}{150994944\pi^{10}} \\
&\quad \left.-\frac{27\left(360\zeta(3)^3-1900\zeta(5)^2-3360\zeta(3)\zeta(7)+4697\zeta(11)\right)}{150994944\pi^{12}}\right)\lambda^7 \\
&+ O(\lambda^8)
\end{aligned}
$$

Strong coupling regime $\lambda \gg 1$

The effect of $k(x)$ change things significantly: strong repulsive central force and attractive 2-body interaction.



Limiting case, $\lambda = \infty$

- distribution $\rho_\infty(x)$ for eigenvalues $x \in (-\infty, \infty)$
- saddle point

$$\fint_{-\infty}^{+\infty} dy\, \rho_\infty(y) \left(\frac{1}{x-y} - K(x-y) \right) = -K(x).$$

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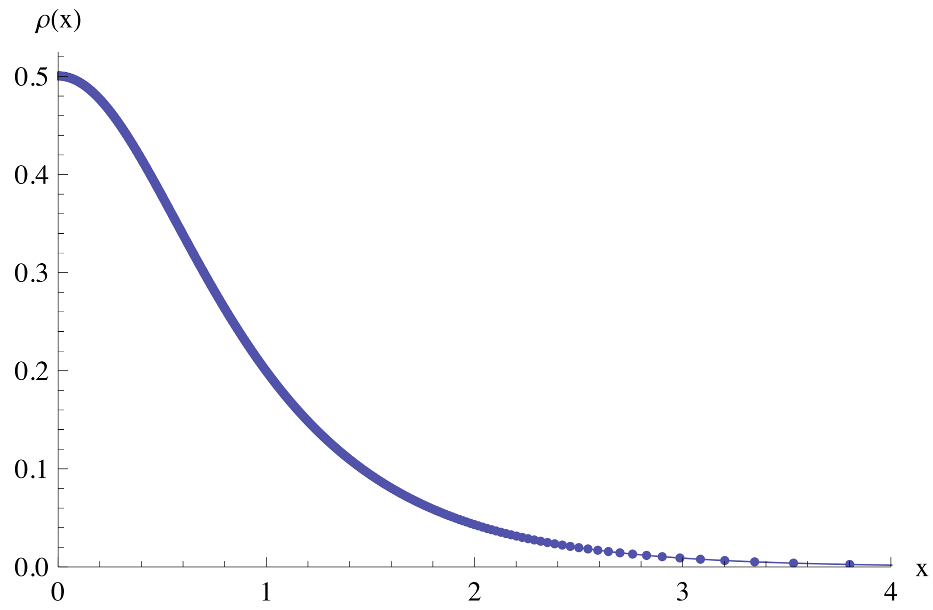
- Fourier transform: $\rho_\infty(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega x} \rho_\infty(\omega)$, $\qquad K(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega x} K(\omega)$

$$\rho_\infty(\omega) = \frac{1}{\cosh \omega} \qquad\qquad \rho_\infty(x) = \frac{1}{2\cosh \frac{\pi x}{2}}$$



$\lambda \gg 1,\ \lambda < +\infty$

- given an observable $\mathcal{O}(x)$, it results $\langle \mathcal{O}(x) \rangle_{\lambda \gg 1} \simeq \int_{-\infty}^{+\infty} dx\, \rho_\infty(x) \mathcal{O}(x)$ only if $\mathcal{O}(x) < e^{\frac{\pi x}{2}}$ at large x

- $\langle W(C) \rangle_{\lambda \gg 1} = \int_{-\mu}^{+\mu} dx\, \rho(x) e^{2\pi x} \simeq e^{2\pi\mu}$, therefore need to compute $\mu = \mu(\lambda)$

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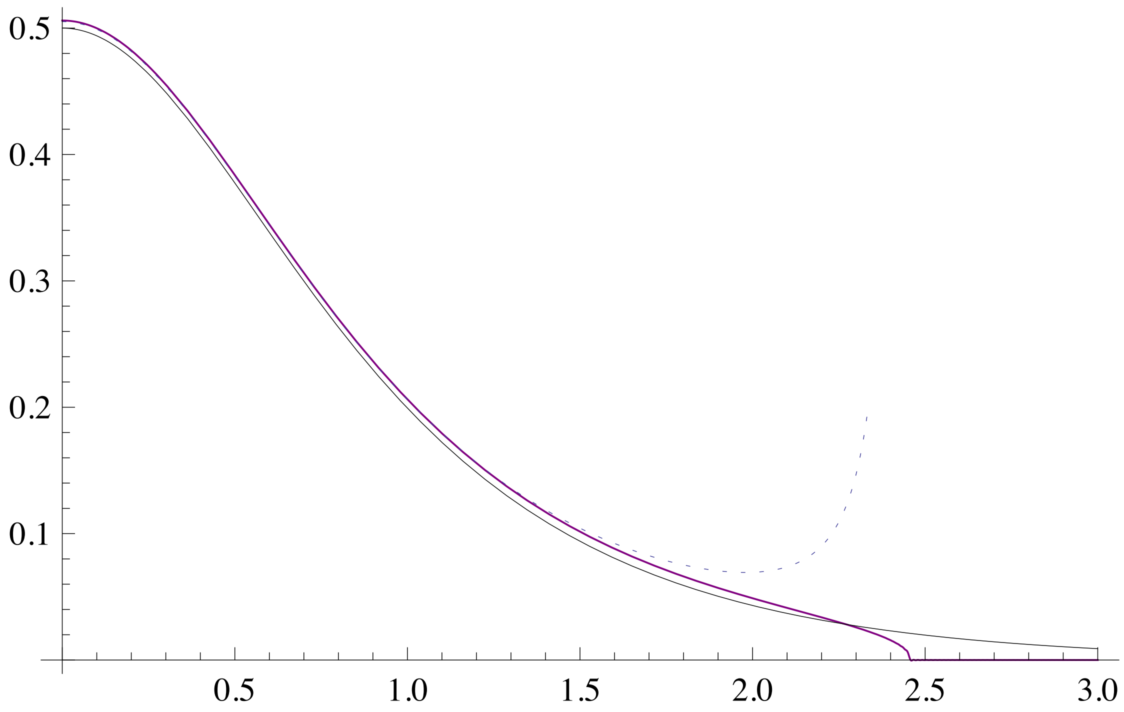
Wiener-Hopf method: generalization of the Fourier transform for the case when an integral equation is defined on a semi-infinite interval

- in the regime $\lambda \gg 1$, the Fourier transform of the distribution $\rho(\omega)$

$$\rho(\omega) = \frac{1}{\cosh\omega} + \frac{2\sinh^2\frac{\omega}{2}}{\cosh\omega} F(\omega) + G_-(\omega)\, \mathrm{e}^{\,i\mu\omega} \sum_{n=0}^{\infty} \frac{r_n\, \mathrm{e}^{\,-\mu\nu_n}}{\omega + i\nu_n} \left(1 - F(-i\nu_n)\right)$$

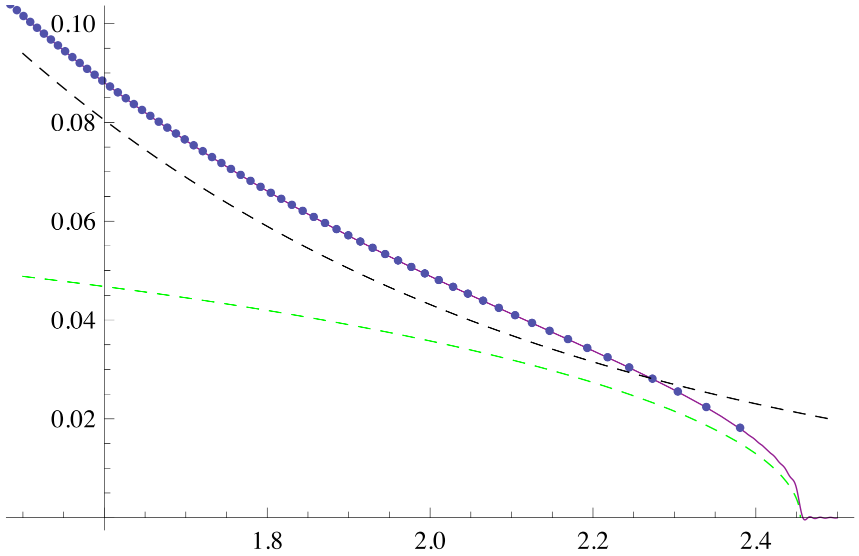


- Wilson loop VEV

$$\langle W(C_{\text{circle}})\rangle_{\lambda\gg 1} = \rho(-2\pi i) = R\frac{\sqrt{\mu}}{\lambda}e^{2\pi\mu}$$

- normalization condition for $\rho(\omega)$ gives

$$C\sqrt{\mu}\,\mathrm{e}^{\pi\mu/2} = \lambda$$



Qualitative estimates: near the endpoint μ the distribution is determined by the attractive force and the repulsive 2-body interaction

$$\Rightarrow \quad \text{Wigner semicircle } \rho(x) \sim \tfrac{8\pi}{\lambda}\sqrt{\mu^2 - x^2}$$

- Wilson loop VEV $\quad \langle W(C_{\text{circle}})\rangle_{\lambda \gg 1} \simeq \frac{8\pi}{\lambda}\int^{\mu} dx\,\sqrt{\mu^2 - x^2}\,\mathrm{e}^{\,2\pi x} = 2\,\frac{\sqrt{\mu}}{\lambda}\,\mathrm{e}^{\,2\pi\mu}$
- normalization condition

$$\int_{\mu}^{\infty} dx\,\rho_\infty(x) \simeq \int_{\mu - z_0}^{\mu} dx\,\frac{8\pi}{\lambda}\sqrt{\mu^2 - x^2} \qquad \to \qquad C\sqrt{\mu}\,\mathrm{e}^{\,\pi\mu/2} = \lambda$$

combining the two expressions

$$\langle W(C_{\rm circle})\rangle_{\lambda\gg 1} = \text{const}\ \frac{\lambda^3}{(\ln\lambda)^{3/2}}$$

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$$\langle W(C_{\text{circle}})\rangle_{\lambda\gg 1} = \text{const}\ \frac{\lambda^3}{(\ln\lambda)^{3/2}}$$

that is equivalent to the string theory prediction

$$\langle W(C_{\text{circle}})\rangle_{\lambda\gg 1} = K T^{-3/2} e^{2\pi T}$$

considering

$$T = \frac{3}{2\pi}\ \ln\lambda$$

Conclusions

- we compute the weak coupling VEV of the Wilson loop in $\mathcal{N} = 2$ SCYM at arbitrary high number of loops

- at strong coupling, the VEV of the loop has a stringy behaviour, although the tension of the string is related to the gauge theory coupling in an unusual way

- the strong coupling result carries information about the unknown string dual

- other interesting probes for the string dual are the Wilson loop in higher rank representation [Fraser, Kumar]

- there are Pestun matrix models for a large class of $\mathcal{N} = 2$ theories. Interesting to study other examples.